Section 2.3 Reducing fractions

In chapter 1, money is used to visualize the expanded form of a number by having all the currency bills limited to place value numbers as shown below.

$1 $10 $100 $1000 …

Money can also be used to visualize fractions. Pennies, nickels, dimes, quarters, and half dollars all represent fractional parts of a dollar as shown below.

Since 100 pennies equal 1 dollar, a penny is one hundredth of a dollar

Since 20 nickels equal 1 dollar, a nickel is one twentieth of a dollar

Since 10 dimes equal 1 dollar, a dime is one tenth of a dollar

Since 4 quarters equal 1 dollar, a quarter is one fourth of a dollar

Since 2 half-dollars equal 1 dollar, a half-dollar is one half of a dollar

Below groups of each of these coins are formed to equal half of a dollar (50 cents) with 50 pennies, 10 nickels, 5 dimes, 2 quarters, or 1 half-dollar coin all equal to half of a dollar (50 cents).

50 pennies written as a fractional part of a dollar	$\$\frac{50}{100}$
10 nickels written as a fractional part of a dollar	$\$\frac{10}{20}$
5 dimes written as a fractional part of a dollar	$\$\frac{5}{10}$
2 quarters written as a fractional part of a dollar	$\$\frac{2}{4}$
1 half-dollar written as a fractional part of a dollar	$\$\frac{1}{2}$

Since the five resulting fractions are all equal to half of a dollar (50 cents) the following fractions are equal to each other.

$$\frac{50}{100} = \frac{10}{20} = \frac{5}{10} = \frac{2}{4} = \frac{1}{2}$$

Equivalent fractions are fractions that are equal to each other.

On the previous page nickels, dimes, quarters, and a half-dollar are used to create the equivalent fractions 10/20, 5/10, 2/4, and 1/2 that are all equal to half of a dollar. Below these fractions are illustrated by shading in appropriate parts of a bar with each bar representing the whole number one. Notice that although on each bar different numbers of parts are shaded, each results with half the bar shaded.

To create an equivalent fraction, multiply both the numerator and denominator of that fraction by the same natural number. Thus each fraction has an infinite number of equivalent fractions.

Example 1 Create two fractions that are equivalent to 3/5

$\frac{3}{5} = \frac{3(\mathbf{2})}{5(\mathbf{2})} = \frac{6}{10}$ Multiply both the numerator and denominator by 2

$\frac{3}{5} = \frac{3(\mathbf{3})}{5(\mathbf{3})} = \frac{9}{15}$ Multiply both the numerator and denominator by 3

6/10 and 9/15 are equivalent to 3/5

Example 2 Create two fractions that are equivalent to 4/7

$\frac{4}{7} = \frac{4(\mathbf{5})}{7(\mathbf{5})} = \frac{20}{35}$ Multiply both the numerator and denominator by 5

$\frac{4}{7} = \frac{4(\mathbf{6})}{7(\mathbf{6})} = \frac{24}{42}$ Multiply both the numerator and denominator by 6

20/35 and 24/42 are equivalent to 4/7

A fraction is written in **reduced form** or **simplest terms** if the numerator and the denominator of the fraction do not share a common factor besides one. In other words, a fraction is written in **reduced form** or **simplest terms** if the only natural number that is divisible into both the numerator and the denominator is one.

Every fraction has a unique reduced fraction form. In this section, three techniques are introduced to write a fraction in simplest term. The first technique shown below involves finding a common factor.

To reduce a fraction by factoring

1) Identify a natural number besides one that is divisible into both the numerator and the denominator. This common divisor is also a common factor.

2) Write both the numerator and denominator in factored form using the common factor.

3) Cancel out the common factor.

4) If the only natural number that is divisible into both the numerator and denominator of the resulting fraction is one then it is in reduced form.

Example 3 Reduce the following fractions: 8/10 15/20

For 8/10 the numerator and denominator are both divisible by 2 so write the both as products with a factor of 2. Then cancel out the common factor 2. The fraction 4/5 is in reduced form since the only common factor of 4 and 5 is one.

$$\frac{8}{10} = \frac{(4)(\mathbf{2})}{(5)(\mathbf{2})} = \frac{(4)(\cancel{\mathbf{2}})}{(5)(\cancel{\mathbf{2}})} = \frac{4}{5}$$

For 15/20 the numerator and denominator are both divisible by 5 so write both as products with a factor of 5. Then cancel out the common factor 5. The fraction 3/4 is in reduced form since the only common factor of 3 and 4 is one.

$$\frac{15}{20} = \frac{(3)(\mathbf{5})}{(4)(\mathbf{5})} = \frac{(3)(\cancel{\mathbf{5}})}{(4)(\cancel{\mathbf{5}})} = \frac{3}{4}$$

Example 4 Reduce the following fractions: 14/63 8/45

For 14/63 the numerator and denominator are both divisible by 7 so write both as products with a factor of 7. Then cancel out the common factor 7. The fraction 2/9 is in reduced form since the only common factor of 2 and 9 is one.

$$\frac{14}{63} = \frac{(2)(\mathbf{7})}{(9)(\mathbf{7})} = \frac{(2)(\cancel{\mathbf{7}})}{(9)(\cancel{\mathbf{7}})} = \frac{2}{9}$$

For 8/45 the only number that divides evenly into both the numerator and denominator is one so it is already in reduced form.

> The **greatest common factor (GCF)** of two counting numbers is the largest natural number which is a factor of both numbers.

When reducing a fraction if the greatest common factor of the numerator and denominator is canceled, the resulting fraction is in reduced form. In the example below fractions are reduced by first writing both the numerator and denominator as products involving their greatest common factor and then canceling out the GCF.

Example 5 Reduce the following fractions: 24/30 16/24

For 24/30 the numerator and denominator are both divisible by 2, 3 and 6 so the greatest common factor is 6. Write both the numerator and the denominators as products with a factor of 6. Then cancel out the greatest common factor 6. The fraction 4/5 is in reduced form since the only common factor of 4 and 5 is one.

$$\frac{24}{30} = \frac{(4)(\mathbf{6})}{(5)(\mathbf{6})} = \frac{(4)(\cancel{\mathbf{6}})}{(5)(\cancel{\mathbf{6}})} = \frac{4}{5}$$

For 16/24 the numerator and denominator are both divisible by 2, 4 and 8 so the great common factor is 8. Write both the numerator and the denominators as products with a factor of 8. Then cancel out the greatest common factor 8. The fraction 2/3 is in reduced form since the only common factor of 2 and 3 is one.

$$\frac{16}{24} = \frac{(2)(\mathbf{8})}{(3)(\mathbf{8})} = \frac{(2)(\cancel{\mathbf{8}})}{(3)(\cancel{\mathbf{8}})} = \frac{2}{3}$$

A second technique, the cancellation method, to reduce a fraction is a shortcut notation in which both the numerator and denominator are both divided by a common divisor with the resulting quotients written above the crossed out numerator and below the crossed out denominator. Below the fraction 15/20 is reduced using both the factoring and cancelation method.

Factoring Method

$$\frac{15}{20} = \frac{(3)(\mathbf{5})}{(4)(\mathbf{5})} = \frac{(3)(\cancel{\mathbf{5}})}{(4)(\cancel{\mathbf{5}})} = \frac{3}{4}$$

Cancellation Method

$$\frac{15}{20} = \frac{\overset{3}{\cancel{15}}}{\underset{4}{\cancel{20}}} = \frac{3}{4}$$

Example 6 Reduce the following fractions using the cancellation method:

18/45 24/32 40/60

For 18/45 the numerator and denominator are both divisible by 3 and 9 so the greatest common factor is 9. Dividing both the numerator and denominator by 9, results in the fraction 2/5 which is in reduced form.

$$\frac{18}{45} = \frac{\overset{2}{\cancel{18}}}{\underset{5}{\cancel{45}}} = \frac{2}{5}$$

For 24/32 the numerator and denominator are both divisible by 2, 4 and 8 so the greatest common factor is 8. Dividing both the numerator and denominator by 8, results in the fraction 3/4 that is in reduced form.

$$\frac{24}{32} = \frac{\overset{3}{\cancel{24}}}{\underset{4}{\cancel{32}}} = \frac{3}{4}$$

For 40/60 the numerator and denominator are both divisible by 10. Dividing both the numerator and denominator by 10, results in the fraction 4/6. The new numerator 4 and denominator 6 are both divisible by 2. Dividing the numerator 4 and denominator 6 by 2, results in the fraction 2/3 which is in reduced form. Note the original fraction 40/60 can be reduced with one cancelation if the greatest common factor 20 is divided into both the original numerator 40 and denominator 60.

$$\frac{40}{60} = \frac{\overset{\overset{2}{\cancel{4}}}{\cancel{40}}}{\underset{\underset{3}{\cancel{6}}}{\cancel{60}}} = \frac{2}{3}$$

A third technique for reducing a fraction is a version of the repeated division process used for prime factorization. This technique is useful when the common factors of the numerators and denominators are not easy to find. The repeated division technique works by starting with the smallest prime that is divisible into both the numerator and denominator and dividing this prime into both the numerator and denominator and writing the resulting quotients on the line below. Continue this process of identifying primes divisible into both numbers on each line until the remaining quotients have only a common factor of one. The product of the prime divisors is the greatest common factor of the original numerator and denominator and the **last quotient is the reduced form of the fraction**.

Example 7 Reduce the fraction 60/96 using the repeated division method.

Both the numerator and denominator 60 and 96 are divisible by the prime 2 with the resulting quotients of 30 and 48. 30 and 48 are also divisible by the prime 2 with the resulting quotients of 15 and 24. 15 and 24 are divisible by the prime 3 with the resulting quotients of 5 and 8. Since 5 and 8 are not divisible by a common prime, the reduced form of the fraction 60/96 is 5/8 (the last quotient) as shown below.

	Num	Den
?	60	96

	Num	Den
2	60	96
?	30	48

	Num	Den
2	60	96
2	30	48
?	15	24

	Num	Den
2	60	96
2	30	48
3	15	24
	5 /	8

To illustrate how this repeated division process works the results are written below in factored form. The greatest common factor of 60 and 96 is (**2**)(**2**)(**3**) which equals 12. Canceling out the greatest common factor 12 written as (**2**)(**2**)(**3**) results in the fraction 5/8 which is in reduced form.

$$\frac{60}{96} = \frac{(\mathbf{2})(\mathbf{2})(\mathbf{3})(5)}{(\mathbf{2})(\mathbf{2})(\mathbf{3})(8)} = \frac{(\cancel{\mathbf{2}})(\cancel{\mathbf{2}})(\cancel{\mathbf{3}})(5)}{(\cancel{\mathbf{2}})(\cancel{\mathbf{2}})(\cancel{\mathbf{3}})(8)} = \frac{5}{8}$$

Students are not expected to write multiple repeated division tables as shown in the solutions to the following problems. Students only need to write the final repeated division table which shows all the prime number divisors and the final quotient line which is the reduced form of the fraction.

Example 8 Reduce the fraction 42/112 using the repeated division method.

Both the numerator and denominator 42 and 112 are divisible by the prime 2 with the resulting quotients of 21 and 56. Both 21 and 56 are divisible by the prime 7 with the resulting quotients of 3 and 8. Since 3 and 8 are not divisible by a common prime, the reduced form of the fraction 42/112 is 3/8 as shown below.

	Num	Den
2	42	112
7	21	56
	3 /	8

To illustrate how this repeated division process works the results are written below in factored form. The greatest common factor of 42 and 112 is (**2**)(**7**) which equals 14. Canceling out the greatest common factor 14 written as (**2**)(**7**) results in the fraction 3/8 which is in reduced form.

$$\frac{42}{112} = \frac{(\mathbf{2})(\mathbf{7})(3)}{(\mathbf{2})(\mathbf{7})(8)} = \frac{(\cancel{\mathbf{2}})(\cancel{\mathbf{7}})(3)}{(\cancel{\mathbf{2}})(\cancel{\mathbf{7}})(8)} = \frac{3}{8}$$

The process of creating an equivalent fraction and reducing a fraction are inverse procedures. To create an equivalent fraction the numerator and denominator are both multiplied by a common number whereas to reduce a fraction the numerator and denominator are both divided by their greatest common factor. To illustrate this inverse relationship an equivalent fraction to 4/7 is created below by multiplying the numerator and denominator by 6 resulting in 24/42, then 24/42 is reduced back to 4/7 by dividing both the numerator and denominator by their greatest common factor 6.

Create a fraction equivalent to 4/7

Multiply both the numerator and denominator by the same number say 6

$$\frac{4}{7} = \frac{4(\mathbf{6})}{7(\mathbf{6})} = \frac{24}{42}$$

Reduce the fraction 24/42

Since GCF of 24 and 42 is 6, divide both numerator and denominator by 6

$$\frac{24}{42} = \frac{(4)(\cancel{\mathbf{6}})}{(7)(\cancel{\mathbf{6}})} = \frac{4}{7}$$

In this section three techniques for reducing or simplifying a fraction are introduced: factoring, cancellation, and repeated division. Most of the numerical fractions in algebra courses involve numerators and denominators whose greatest common factors are easily identified by mental calculations and which are simplified using either the cancelation or factoring method. The repeated division technique is useful for fractions whose numerator and denominators have a greatest common factor which is not easily identified without doing written calculations. Below as a summary the fraction 30/45 is reduced using the three techniques.

Factoring Method

$$\frac{30}{45} = \frac{(2)(\mathbf{15})}{(3)(\mathbf{15})} = \frac{(2)(\cancel{\mathbf{15}})}{(3)(\cancel{\mathbf{15}})} = \frac{2}{3}$$

Cancellation Method

$$\frac{30}{45} = \frac{\overset{\overset{2}{\cancel{6}}}{\cancel{30}}}{\underset{\underset{3}{\cancel{9}}}{\cancel{45}}} = \frac{2}{3}$$

Repeated Division Method

	Num	Den
3	30	45
5	10	15
	2 /	3

$$\frac{30}{45} = \frac{(\mathbf{3})(\mathbf{5})(2)}{(\mathbf{3})(\mathbf{5})(3)} = \frac{(\cancel{\mathbf{3}})(\cancel{\mathbf{5}})(2)}{(\cancel{\mathbf{3}})(\cancel{\mathbf{5}})(3)} = \frac{2}{3}$$

Exercises 2.3

1-6. Create two fractions that are equivalent to the following fractions.

1. $\frac{5}{6}$
2. $\frac{3}{4}$
3. $\frac{1}{7}$
4. $\frac{3}{8}$
5. $\frac{7}{10}$
6. $\frac{8}{9}$

7-15. Reduce the following fractions using **the factoring method**.

7. $\frac{14}{16}$
8. $\frac{9}{12}$
9. $\frac{5}{30}$
10. $\frac{8}{24}$
11. $\frac{28}{42}$
12. $\frac{15}{25}$
13. $\frac{18}{24}$
14. $\frac{36}{48}$
15. $\frac{30}{45}$

16-24. Reduce the following fractions using **the cancellation method**.

16. $\frac{8}{10}$
17. $\frac{18}{21}$
18. $\frac{25}{45}$
19. $\frac{6}{30}$
20. $\frac{32}{48}$
21. $\frac{21}{49}$
22. $\frac{8}{32}$
23. $\frac{26}{39}$
24. $\frac{36}{54}$

25-30. Reduce the following fractions using **the repeated division method**.

25. $\frac{28}{36}$
26. $\frac{42}{72}$
27. $\frac{54}{90}$
28. $\frac{60}{96}$
29. $\frac{120}{135}$
30. $\frac{66}{110}$