Section 2.1 Divisibility tests, factors, and primes

The natural numbers are the whole numbers excluding zero. In this section divisors are limited to natural numbers to avoid divisions by zero which are undefined.

> The **natural numbers** { 1, 2, 3, 4, 5, 6, 7,… } are the numbers which are used for counting. The natural numbers is another name for the **counting numbers**.

A natural number is **divisible** by a divisor if that divisor divides evenly into that number with the resulting quotient having no remainder. There is a simple test which determines whether a counting number is divisible by 2, 5 and 10. Below is a list of multiples of 2, 5 and 10. Look at each individual list and try to find a pattern.

2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34, 36, 38, …

The multiples of 2 are even numbers with a 0, 2, 4, 6, or 8 in the ones place

5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60, 65, 70, 75, 80, 85, 90, …

The multiples of 5 have a 0 or 5 as the last digit (the digit in the ones place)

10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, 120, 130, 140, 150, 160, …

The multiples of 10 have a 0 as the last digit (the digit in the ones place)

There is a test that determines whether a natural number is divisible by 3 but it is more complicated than simply checking the last digit. For multiples of 3 the sum of the digits is also divisible by 3. Below is a list of multiples of 3.

3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, 42, 45, 48, 51, …

For 12, 1 + 2 = 3 is divisible by 3

For 15, 1 + 5 = 6 is divisible by 3

For 18, 1 + 8 = 9 is divisible by 3

For 21, 2 + 1 = 3 is divisible by 3

For 24, 2 + 4 = 6 is divisible by 3

For 27, 2 + 7 = 9 is divisible by 3

For 30, 3 + 0 = 3 is divisible by 3

For 33, 3 + 3 = 6 is divisible by 3

For 36, 3 + 6 = 9 is divisible by 3

For 39, 3 + 9 = 12 is divisible by 3

For 42, 4 + 2 = 6 is divisible by 3

For 45, 4 + 5 = 9 is divisible by 3

For 48, 4 + 8 = 12 is divisible by 3

For 51, 5 + 1 = 6 is divisible by 3

The observations about the multiples of 2, 3, 5 and 10 are the basis for the following divisibility test. The divisibility test for 2, 5 and 10 are based on the last digit while the divisibility test for 3 involves the sum of the digits.

Divisibility test for 2, 3, 5, and 10 divisors

A natural number is divisible by 2 if it is even with a last digit of 0, 2, 4, 6 or 8

A natural number is divisible by 3 if the sum of its digits is divisible by 3

A natural number is divisible by 5 if it has a last digit of 0 or 5

A natural number is divisible by 10 if it has a last digit of 0

Example 1 Determine if the following numbers are divisible by 2, 3, 5 or 10

42 75 5730 12,047

42 is divisible by 2 and 3

Divisible by 2 since 42 is even
Divisible by 3 since 4 + 2 equals 6 which is divisible by 3
Not divisible by 5 since the last digit 2 is neither 0 nor 5
Not divisible by 10 since the last digit 2 is not 0

75 is divisible by 3 and 5

Not divisible by 2 since 75 is not even
Divisible by 3 since 7 + 5 equals 12 which is divisible by 3
Divisible by 5 since the last digit is 5
Not divisible by 10 since the last digit 5 is not 0

5730 is divisible by 2, 3, 5 and 10

Divisible by 2 since 5,730 is even
Divisible by 3 since 5 + 7 + 3 + 0 equals 15 which is divisible by 3
Divisible by 5 since the last digit is 0
Divisible by 10 since the last digit is 0

12,047 is **not** divisible by 2, 3, 5 or 10

Not divisible by 2 since 12,047 is not even
Not divisible by 3 since 1 + 2 + 0 + 4 + 7 equals 14 which is not divisible by 3
Not divisible by 5 since the last digit 7 is neither 0 nor 5
Not divisible by 10 since the last digit 7 is not 0

If a natural number is divisible by a divisor, than that divisor is also a factor of that number. Consider 15 which is divisible only by 1, 3, 5, and 15. Since the only combinations of counting numbers that when multiplied equal 15 are $1 \cdot 15 = 15$ and $3 \cdot 5 = 15$, the factors of 15 are also 1, 3, 5, and 15. So the list of all divisors that divide evenly into a number is also the list of all the factors of that number.

Example 2 List all the factors of 18

Since 18 is divisible by 1 with $1 \cdot 18 = 18$, list 1 and 18	{ 1, 18 }
Since 18 is divisible by 2 with $2 \cdot 9 = 18$, list 2 and 9	{ 1, 2, 9, 18 }
Since 18 is divisible by 3 with $3 \cdot 6 = 18$, list 3 and 6	{ 1, 2, 3, 6, 9, 18 }

18 is not divisible by 4 or 5 and the factor 6 is already listed

The set { 1, 2, 3, 6, 9, 18 } lists all the factors of 18

Example 3 List all the factors of 35

Since 35 is divisible by 1 with $1 \cdot 35 = 35$, list 1 and 35	{ 1, 35 }
35 is not divisible by 2, 3 or 4	
Since 35 is divisible by 5 with $5 \cdot 7 = 35$, list 5 and 7	{ 1, 5, 7, 35 }

35 is not divisible by 6 and the factor 7 is already listed

The set { 1, 5, 7, 35 } lists all the factors of 35

Example 4 List all the factors of 24

Since 24 is divisible by 1 with $1 \cdot 24 = 24$, list 1 and 24	{ 1, 24 }
Since 24 is divisible by 2 with $2 \cdot 12 = 24$, list 2 and 12	{ 1, 2, 12, 24 }
Since 24 is divisible by 3 with $3 \cdot 8 = 24$, list 3 and 8	{ 1, 2, 3, 8, 12, 24 }
Since 24 is divisible by 4 with $4 \cdot 6 = 24$, list 4 and 6	{ 1, 2, 3, 4, 6, 8, 12, 24 }

24 is not divisible by 5 and the factor 6 is already listed

The set { 1, 2, 3, 4, 6, 8, 12, 24 } lists all the factors of 20

Example 5 List all the factors of 17

Since 17 is divisible by 1 with $1 \cdot 17 = 17$, list 1 and 17	{ 1, 17 }

17 is a prime number which is divisible only by 1 and itself.

The set { 1, 17 } lists are all the factors of 17

Below is a table that lists all the counting numbers from 2 to 50. Start by bolding the first number 2 and cross out all the numbers divisible by 2. This results with all the even numbers besides 2 being crossed out as shown below.

	2	3	~~4~~	5	~~6~~	7	~~8~~	9	~~10~~
11	~~12~~	13	~~14~~	15	~~16~~	17	~~18~~	19	~~20~~
21	~~22~~	23	~~24~~	25	~~26~~	27	~~28~~	29	~~30~~
31	~~32~~	33	~~34~~	35	~~36~~	37	~~38~~	39	~~40~~
41	~~42~~	43	~~44~~	45	~~46~~	47	~~48~~	49	~~50~~

Now bold the next number 3 and cross out all the numbers divisible by 3 that still remain on the table as shown below with 9, 15, 21, 27, 33, 39 and 45 crossed out.

	2	**3**	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	25	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	35	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	49	~~50~~

Now bold the next uncrossed number 5 and cross out all the numbers divisible by 5 that still remain on the table as shown below with 25 and 35 crossed out

	2	**3**	~~4~~	**5**	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	49	~~50~~

Now bold the next uncrossed number 7 and cross out all the numbers divisible by 7 that still remain on the table as shown below with 49 crossed out. Then bold all the remaining numbers that are not crossed out. Do you recognize the bolded numbers?

	2	**3**	~~4~~	**5**	~~6~~	**7**	~~8~~	~~9~~	~~10~~
11	~~12~~	**13**	~~14~~	~~15~~	~~16~~	**17**	~~18~~	**19**	~~20~~
~~21~~	~~22~~	**23**	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	**29**	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	**37**	~~38~~	~~39~~	~~40~~
41	~~42~~	**43**	~~44~~	~~45~~	~~46~~	**47**	~~48~~	~~49~~	~~50~~

The bolded numbers in the table 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47 are the first fifteen prime numbers. The simple and elegant algorithm performed on the previous page is called the **sieve of Eratosthenes** after the famous Greek mathematician who was born in 276 BC. There is no largest prime number, so even today all the prime numbers have not been found.

Prime numbers are natural numbers which are divisible by only two different natural numbers, 1 and itself. Prime numbers only have two distinct factors, 1 and itself.

Composite numbers are natural numbers which are divisible by at least one natural number besides 1 and itself. Composite numbers have more than two distinct factors.

All natural numbers are either composite or prime numbers except the number 1 which is neither since it only divisible by itself. The number 2 is the only even prime number since all other even numbers are divisible by 2.

Example 6 Determine whether 85, 47, and 291 are prime or composite.

85 is a composite number.
85 is divisible by 5 since its last digit is 5, with the product $5 \cdot 17$ equaling 85.

47 is a prime number.
47 is divisible only by 1 and 47.

291 is a composite number.
291 is divisible by 3 since the sum 2 + 9 + 1 equals 12 which is divisible by 3 with the product $3 \cdot 97$ equaling 291

The **fundamental theorem of arithmetic** states that every composite number can be written as a product in a unique way using only prime number factors. Below is the prime factorization for the first eight composite numbers.

$4 = (2)(2) = (2)^2$ $\qquad$ $10 = (2)(5)$

$6 = (2)(3)$ $\qquad$ $12 = (2)(2)(3) = (2)^2(3)$

$8 = (2)(2)(2) = (2)^3$ $\qquad$ $14 = (2)(7)$

$9 = (3)(3) = (3)^2$ $\qquad$ $15 = (3)(5)$

> **Prime factorization** is the process of writing a composite number as a product consisting of only prime factors.

One technique to write composite number in prime factorization format is **repeated division using prime divisors**. This process divides the composite number by all its divisible prime divisors starting with the smallest prime divisor until the remaining quotient is a prime number. To show this process the division box is drawn upside down with the quotient on the bottom instead of the top. To use repeated division to write 42 in prime factorization format as shown below divide by smallest prime number 2. The resulting quotient 21 is not divisible by 2, so try the next prime number 3 as a divisor. 21 is divisible by 3 with the quotient 7. Once the quotient from a division is prime stop the process, with 42 = (2)(3)(7)

$$\begin{array}{r|l} \mathbf{?} & 4\,2 \\ \cline{2-2} \end{array} \qquad \Bigg| \qquad \begin{array}{r|l} \mathbf{2} & 4\,2 \\ \cline{2-2} \mathbf{?} & 2\,1 \\ \cline{2-2} \end{array} \qquad \Bigg| \qquad \begin{array}{r|l} \mathbf{2} & 4\,2 \\ \cline{2-2} \mathbf{3} & 2\,1 \\ \cline{2-2} & \;\;\mathbf{7} \end{array}$$

To use repeated division to write 90 in primed factorization format as shown below divide by smallest prime number 2. The resulting quotient 45 is not divisible by 2, so try the next prime number 3 as a divisor. 45 is divisible by 3 with the quotient 15. 15 is also divisible by 3 which results with quotient 5. Once the quotient from a division is prime stop the process, with 90 = (2)(3)(3)(5) = $(2)(3)^2(5)$

$$\begin{array}{r|l} \mathbf{?} & 9\,0 \\ \cline{2-2} \end{array} \qquad \Bigg| \qquad \begin{array}{r|l} \mathbf{2} & 9\,0 \\ \cline{2-2} \mathbf{?} & 4\,5 \\ \cline{2-2} \end{array} \qquad \Bigg| \qquad \begin{array}{r|l} \mathbf{2} & 9\,0 \\ \cline{2-2} \mathbf{3} & 4\,5 \\ \cline{2-2} \mathbf{?} & 1\,5 \\ \cline{2-2} \end{array} \qquad \Bigg| \qquad \begin{array}{r|l} \mathbf{2} & 9\,0 \\ \cline{2-2} \mathbf{3} & 4\,5 \\ \cline{2-2} \mathbf{3} & 1\,5 \\ \cline{2-2} & \;\;\mathbf{5} \end{array}$$

To use repeated division to write 77 in primed factorization format since 77 is not divisible by the prime numbers 2, 3 or 5 divide by the next prime 7 as shown below. Since the resulting quotient 11 is prime stop the process, with 77 = (7)(11)

$$\begin{array}{r|l} \mathbf{?} & 7\,7 \\ \cline{2-2} \end{array} \qquad \Bigg| \qquad \begin{array}{r|l} \mathbf{7} & 7\,7 \\ \cline{2-2} & \mathbf{1\,1} \end{array}$$

Students are not expected to write multiple repeated division tables as shown in the solutions to the following problems. These are displayed to illustrate how the process works. Students only need to write the final repeated division table which shows all the prime number divisors.

Example 7 Write 63 in prime factorization format.

$63 = (3)(3)(7) = (3)^2(7)$

63 is not divisible by 2 (not even) but it is divisible by 3 since sum of its digits is divisible by 3. The quotient 21 is also divisible by 3 and the resulting quotient 7 is a prime number as shown below.

$$\begin{array}{r|l} \mathbf{?} & 6\ 3 \\ \hline \end{array} \qquad \begin{array}{r|l} \mathbf{3} & 6\ 3 \\ \hline \mathbf{?} & 2\ 1 \\ \hline \end{array} \qquad \begin{array}{r|l} \mathbf{3} & 6\ 3 \\ \hline \mathbf{3} & 2\ 1 \\ \hline & \ \ \ \mathbf{7} \end{array}$$

Example 8 Write 130 in prime factorization format.

$130 = (2)(5)(13)$

130 is divisible by 2 with the resulting quotient 65. The quotient 65 is not divisible by 2 or 3 but is divisible by 5 and the resulting quotient 13 is a prime number as shown below.

$$\begin{array}{r|r} \mathbf{?} & 1\ 3\ 0 \\ \hline \end{array} \qquad \begin{array}{r|r} \mathbf{2} & 1\ 3\ 0 \\ \hline \mathbf{?} & 6\ 5 \\ \hline \end{array} \qquad \begin{array}{r|r} \mathbf{2} & 1\ 3\ 0 \\ \hline \mathbf{5} & 6\ 5 \\ \hline & \mathbf{1\ 3} \end{array}$$

Prime factorization is used to reduce fractions, to find the least common multiple of two or more counting numbers, and to find the greatest common factor of two or more counting numbers in later sections.

Exercises 2.1

1. Define the composite numbers.
2. Define the prime numbers.
3. List the first fifteen prime numbers.

4-9 Determine if the following numbers are divisible by 2, 3, 5 or 10

4.	62	5.	75	6.	171
7.	200	8.	720	9.	161

10-15 Determine whether the following numbers are composite or prime.

10.	48	11.	43	12.	63
13.	81	14.	57	15.	59

16-21 List all the factors of the following numbers.

16.	16	17.	18	18.	32
19.	40	20.	60	21.	100

22-33 Use repeated division to write the following composite numbers in prime factorization format.

22.	20	23.	16	24.	24
25.	32	26.	100	27.	75
28.	48	29.	140	30.	78
31.	165	32.	102	33.	280

34 Apply the sieve of Eratosthenes to find all the prime numbers less than 100 using the table on the next page.

	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	84	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100