Section 1.8 Exponents, square roots, and the order of operations

Multiplication is the arithmetic operation that determines the total resulting from repeatedly adding the same number. For instance, 4 groups of 9 written in expanded addition form 9 + 9 + 9 + 9 can be written in a compact notation using multiplication as 4×9. In the same way that multiplication serves as the notation for repeated addition of the same number, exponential notation serves as the notation for repeated multiplication by the same number. In exponential notation format $(base)^{power}$, the **base** represents the number that is multiplied repeatedly and the **exponent** or **power** indicates how many times the base appears as a factor in the product. For instance, three raised to the fifth power is written in exponential notation with base 3 and power 5 as 3^5 which in expanded form is equal to $3 \times 3 \times 3 \times 3 \times 3$. Since a parenthesis or a dot is also used to indicate multiplication, 3^5 can also be written in expanded form as (3)(3)(3)(3)(3) or $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$

> The **exponential notation** b^n is the product formed by listing the base b as a factor n times as shown below.
>
> $$b^n = \underbrace{b \times b \times \ldots \times b}_{n\ times}$$

Example 1 Write the following in expanded form: 5^4 6^3

$5^4 = 5 \times 5 \times 5 \times 5$
5 raised to the fourth power
Base is 5 and exponent is 4

$6^3 = 6 \cdot 6 \cdot 6$
6 raised to the third power
Base is 6 and exponent is 3

> A number raised to the second power is read as the number **squared.**
>
> A number raised to the third power is read as the number **cubed.**

Example 2 Write the following products in exponential notation:

$7 \times 7 \times 7 = 7^3$

$(12)(12) = 12^2$

$4 \cdot 4 \cdot 4 \cdot 9 \cdot 9 = (4)^3(9)^2$

Example 3 Evaluate the following:

5 squared

$5^2 = 5 \times 5 = 25$

2 raised to fourth power

$2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$

10 cubed

$10^3 = (1\mathbf{0})(1\mathbf{0})(1\mathbf{0}) = 1\mathbf{000}$

A **perfect square** is a number that results from a product formed by raising a number to the second power. Below the perfect squares 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196 and 225 are generated by squaring the first 15 counting numbers.

$1^2 = \mathbf{1}$	$2^2 = \mathbf{4}$	$3^2 = \mathbf{9}$	$4^2 = \mathbf{16}$	$5^2 = \mathbf{25}$
$6^2 = \mathbf{36}$	$7^2 = \mathbf{49}$	$8^2 = \mathbf{64}$	$9^2 = \mathbf{81}$	$10^2 = \mathbf{100}$
$11^2 = \mathbf{121}$	$12^2 = \mathbf{144}$	$13^2 = \mathbf{169}$	$14^2 = \mathbf{196}$	$15^2 = \mathbf{225}$

The **radicand** of a square root is the number that is inside the square root symbol, the radicand of $\sqrt{A}$ is A.

The **square root** is the number that when squared (multiplied times itself) equals the radicand of the square root.

Below the square roots whose radicands are the first fifteen perfect square numbers 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196 and 225 are listed. In this textbook, the radicands of the square roots will be perfect squares. Square roots with radicands that are not perfect squares are covered in a pre-algebra course.

$\sqrt{1} = 1$	$\sqrt{4} = 2$	$\sqrt{9} = 3$	$\sqrt{16} = 4$	$\sqrt{25} = 5$
$\sqrt{36} = 6$	$\sqrt{49} = 7$	$\sqrt{64} = 8$	$\sqrt{81} = 9$	$\sqrt{100} = 10$
$\sqrt{121} = 11$	$\sqrt{144} = 12$	$\sqrt{169} = 13$	$\sqrt{196} = 14$	$\sqrt{225} = 15$

Example 4 Evaluate the following square roots.

$\sqrt{16} = 4$	$\sqrt{36} = 6$	$\sqrt{100} = 10$
Square root of 16 is 4	Square root of 36 is 6	Square root of 100 is 10

The four basic arithmetic operations addition, subtraction, multiplication, and division as well as exponential notation and square roots have been defined in this chapter. Now, a rule is needed which gives the order in which operations are performed to insure that a numerical expression that contains multiple operations results in only one numerical value when evaluated. Consider the numerical expression 5 + 2(3) with an addition and a multiplication operation. If there is no rule to determine which operation to perform first, different answers will result as shown below depending on the order in which the operations are done. When the multiplication operation is done first which is the correct way the result is 11, but if the addition operation is performed first the result is 21.

multiplication operation is done first	addition operation is done first
☑ $5 + \mathbf{2(3)} = \mathbf{5 + 6} = 11$ ☑	✗ $\mathbf{5 + 2}(3) = \mathbf{7(3)} = 21$ ✗

To avoid the above situation, a rule that describes the order in which operations are performed is necessary. The order of operations listed below serve as the rules that ensure that a numerical expression when evaluated will result in one unique answer. Try to memorize the order of operations since these rules are used not only in this arithmetic class but in future algebra and math courses.

The order of operations

1. Do all the operations inside grouping symbols such as **parenthesis** or brackets.
2. Evaluate all **exponents** and square roots.
3. Do all the **multiplications** and **divisions** as they occur from left to right.
4. Do all the **additions** and **subtractions** as they occur from left to right.

Try to memorize the order of operations. As a memorization tool, the first letters of the bolded works **p**arenthesis, **e**xponents, **m**ultiplication, **d**ivision, **a**ddition, and **s**ubtraction in the order of operations can be combined to form **PEMDAS** which can be memorized by the saying “Please Excuse My Dear Aunt Sally”. Be careful using PEMDAS since multiplication is not always done before division with the order in which they are performed determined by which operation occurs first in the expression as read from left to right. Similarly using PEMDAS addition is not always done before subtraction with the order in which they are done determined by which operation occurs first in the expression as read from left to right.

Returning to the numerical expression 5 + 2·3 since it does not have operations inside grouping symbols or exponents when evaluating this expression start at step three and perform the multiplication operation which simplifies the numerical expression to 5 + 6. Next, go to step four and add these numbers so that the correct value of this numerical expression is 11 as shown below.

5 + **2(3)**	Perform multiplication (Step 3)
= **5 + 6**	Perform addition (Step 4)
= 11	

If the above numerical expression is written using with the addition operation inside parenthesis as (5 + 2)(3), when evaluating this expression first start at step one and evaluate the addition operation inside the grouping symbol which simplifies to the numerical expression (7)(3) then multiply to obtain 21 as shown below.

(**5 + 2**)(3)	Evaluate addition inside parenthesis (Step 1)
= (7)(3)	Perform multiplication (Step 3)
= 21	

Example 5 Evaluate the following numerical expressions. Show all steps.

12 – 5 + 3	
= 12 – 5 + 3	Subtraction occurs to the left of the addition (Step 4)
= **7 + 3**	Now add (Step 4)
= 10	

2(3) + 4(5)	
= 2(3) + **4(5)**	Do the multiplications (Step 3)
= **6 + 20**	Now add (Step 4)
= 26	

$(2)^3(5)^2$	
= $\mathbf{(2)^3(5)^2}$	Evaluate the exponents (Step 2)
= (**8)(25)**	Do the multiplications (Step 3)
= 200	

Example 6 Evaluate the following numerical expressions. Show all steps.

$16 \div 2(4)$

$= \mathbf{16 \div 2}(4)$ Division occurs to the left of multiplication (Step 3)

$= \mathbf{8(4)}$ Do the multiplication (Step 3)

$= 32$

$5^2 - 4(3) - 7$

$= \mathbf{5^2} - 4(3) - 7$ Evaluate the exponent (Step 2)

$= 25 - \mathbf{4(3)} - 7$ Do the multiplication (Step 3)

$= \mathbf{25 - 12} - 7$ Do the subtraction on the left first (Step 4)

$= 13 - 7$ Do the next subtraction (Step 4)

$= 6$

$2^3 - 7 + \sqrt{81}$

$= \mathbf{2^3} - 7 + \sqrt{\mathbf{81}}$ Evaluate the exponent and square root (Step 2)

$= \mathbf{8 - 7} + 9$ Subtraction occurs to the left of the addition (Step 4)

$= \mathbf{1 + 9}$ Do the addition (Step 4)

$= 10$

$5\sqrt{36} + 3(4) - \frac{12}{2}$

$= 5\sqrt{\mathbf{36}} + 3(4) - \frac{12}{2}$ Evaluate the square root (Step 2)

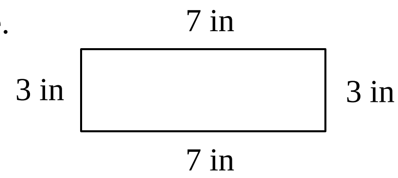
$= \mathbf{5 \cdot 6} + \mathbf{3(4)} - \mathbf{\frac{12}{2}}$ Do multiplications and division left to right (Step 3)

$= \mathbf{30 + 12} - 6$ Addition occurs to the left of subtraction (Step 4)

$= \mathbf{42} - \mathbf{6}$ Do the subtraction (Step 4)

$= 36$

Example 7 Find the perimeter of the following rectangle.



To find the perimeter of this rectangle find the sum of the length of the four sides. The perimeter of this rectangle is 20 inches as shown below

$$3 + 7 + 3 + 7 = (3 + 7) + (3 + 7) = 10 + 10 = 20$$

For rectangles since the opposite sides are the same, the perimeter can be found by doubling the length and the width separately and then adding the resulting products.

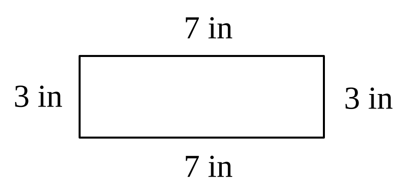
> The **perimeter of a rectangle** is the sum of twice the length and twice the width.
>
> 2(length) + 2(width)

Example 8 Find the perimeter of the rectangle using the perimeter formula.

2(7 in) + 2(3 in)

= 14 in + 6 in Do the multiplication (Step 3)

= 20 in Do the addition (Step 4)



In the order of operations the first step is to evaluate all operations inside parenthesis or grouping symbols. This allows the author of a numerical expression to put an operation with a low order priority such as addition from step 4 inside the parenthesis where it evaluated first with high order priority. To find the mean of 7, 8 and 12 the numbers are added and then divided by three. Which of the following numerical expressions 7 + 8 + 12 ÷ 3 or (7 + 8 + 12) ÷ 3 calculates the mean? The first expression 7 + 8 + 12 ÷ 3 does not calculate the mean, since the division of 12 by 3 is done first then the number are added. The second expression (7 + 8 + 12) ÷ 3 does calculate the mean since the parenthesis indicates that the addition of the numbers is done first then the division by three.

☒ **Not** the mean of 7, 8 & 12 ☒	☑ The **mean** of 7, 8 & 12 ☑
Divide first then add	Add first then divide
7 + 8 + **12 ÷ 3**	(**7 + 8 + 12**) ÷ 3
= **7 + 8 + 4**	= **27 ÷ 3**
= 19	= 9

The following problems have numerical expressions with operations inside grouping symbols, so the first step is to evaluate the operations inside these grouping symbols.

Example 9 Evaluate the following numerical expressions. Show all steps.

2(3 + 8)

= 2(**3 + 8**)	Do the addition inside parenthesis (Step 1)
= **2(11)**	Do the multiplication (Step 3)
= 22	

69 – 5(7 + 3)

= 69 – 5(**7 + 3**)	Do the addition inside parenthesis (Step 1)
= 69 – **5(10)**	Do the multiplication (Step 3)
= **69 – 50**	Do the subtraction (Step 4)
= 19	

20 ÷ 5(6 – 4)

= 20 ÷ 5(**6 – 4**)	Do the subtraction inside parenthesis (Step 1)
= **20 ÷ 5**(2)	Division is to the left of the multiplication (Step 3)
= **4(2)**	Do the multiplication (Step 3)
= 8	

40 + 5(9 – 3) – 12

= 40 + 5(**9 – 3**) – 12	Do the subtraction inside parenthesis (Step 1)
= 40 + **5(6)** – 12	Do the multiplication (Step 3)
= **40 + 30** – 12	Addition occurs to the left of subtraction (Step 4)
= **70 – 12**	Do the subtraction (Step 4)
= 58	

Exercises 1.8

1-4 Write the following in expanded form.

1. 4×7
2. 3×8
3. 9^4
4. 6^3

5-14 Write the following products in exponential notation.

5. $8 \times 8 \times 8 \times 8 \times 8$
6. $5 \times 5 \times 5 \times 5$
7. (2)(2)(2)(2)(2)
8. (8)(8)(8)
9. Six raised to the eighth power
10. Three raised to the fourth power
11. Eight squared
12. Four cubed
13. (4)(4)(5)(5)(5)
14. $7 \times 7 \times 7 \times 9 \times 9 \times 9 \times 9$

15-20 Evaluate the following expressions.

15. 3^3
16. 2^4
17. 8^2
18. 5^3
19. $(2)^3(3)^2$
20. $(4)^2(5)^2$

21-26 Evaluate the following square roots.

21. $\sqrt{81}$
22. $\sqrt{36}$
23. $\sqrt{9}$
24. $\sqrt{64}$
25. $\sqrt{121}$
26. $\sqrt{144}$

27-50 Evaluate the following numerical expressions. Show all steps.

27. $4 + 5(6)$
28. $22 - 4(5)$
29. $13 - 5 + 4$
30. $2^4 - 7 + 11$
31. $5(3) + 2(7)$
32. $4(7) - 5(3)$
33. $5(4) - \frac{32}{8} - 2$
34. $\frac{48}{6} - \frac{20}{4} + 3(1)$
35. $12 + 6 \div 3$
36. $4 \cdot 6 \div 3$

37. $2^2 + 5^2 - \sqrt{16}$

38. $12 + 6\sqrt{25} - 40$

39. $5(3 - 1) - 7$

40. $3(4 + 5) - 4(3 + 2)$

41. $4 \cdot 5 + 3^3 - 18$

42. $5 + 2(4 - 1) - 11$

43. $3 + 4 + 5(1 + 2)$

44. $1^2 + 2^2 + 3^2 + 4^2$

45. $(1 + 2 + 3 + 4)^2$

46. $20 + 2(3 + 4)$

47. $4 + 2 \cdot (7 - 5)^3$

48. $(2 + 5)^2$

49. $4 + 8 \div 4 \cdot 2 - 5$

50. $100 - 2 \cdot (3 + 4)^2$

51-56 Find a numerical expression that models the following problems. Evaluate the numerical expression and answer the question.

51. Find the mean of the following ages: 23, 18, 28, 22 and 19

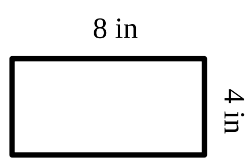
52. Find the mean of the following prices: \$120, \$145, \$132 and \$123

53. Find the mean size of the following TV screen sizes: 34 inches, 36 inches and 41 inches

54. Find the perimeter of a rectangle whose length is 5 feet and width is 3 feet.

55-56 Find the perimeter of the rectangles which are drawn below.

55.



56.