Section 4.1 Exponents & Logarithms

Exponential notation serves as a shorthand notation for products formed by repeated multiplication of the same number. For instance, the product of ten times ten times ten times ten written (10)(10)(10)(10) is written in exponential notation as ten raised to the fourth power $(10)^4$ with base 10 and exponent 4.

Notation The **exponential notation** $(a)^n$ is the product formed by listing the base a as a factor n times.

$$(a)^n = \underbrace{a \times a \times \ldots \times a}_{n\ \ times}$$

Example 1 Write the following in expanded form: $(5)^4$ $(6)^3$

$(5)^4 = 5 \times 5 \times 5 \times 5$

5 raised to the fourth power
Base is 5 and exponent is 4

$(6)^3 = (6)(6)(6)$

6 raised to the third power
Base is 6 and exponent is 3

Example 2 Evaluate the following:

5 squared

$(5)^2 = 5 \times 5 = 25$

2 raised to fourth power

$(2)^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$

10 cubed

$(10)^3 = (1\mathbf{0})(1\mathbf{0})(1\mathbf{0}) = 1\mathbf{000}$

Definition The product of a number and its **reciprocal** is one. All non-zero real numbers have reciprocals.

Example 3 Find the reciprocals of 10, -6, and 2/3

The reciprocal of 10 is 1/10 since (10)(1/10) = 1

The reciprocal of -6 is -1/6 since (-6)(-1/6) = 1

The reciprocal of 2/3 is 3/2 since (2/3)(3/2) = 1

Notation The mathematical notation that indicates a reciprocal of a number is to raise that number to the negative one power. Thus, a^{-1} denotes the reciprocal of the base a with $(a)^{-1} = 1/a$

For numbers, the negative of a number indicates the opposite of that number. The opposite of a number is the number that is the same distance from zero on a number line but on the opposite side of the number line. The sum of a number and its opposite always equals zero. For example, the opposite of three is negative three and the sum of three and negative three is zero. Do not confuse negative exponents with the opposite of a number! Negative exponents indicate the reciprocal of a number. The product of a quantity and its reciprocal is equal to one.

Example 4 Simplify the following:

$(10)^{-1} = 1/10$	Reciprocal of 10 is 1/10
$(2/3)^{-1} = 3/2$	Reciprocal of 2/3 is 3/2
$(3x)^{-1} = 1/(3x)$	Reciprocal of $3x$ is $1/(3x)$
$3x^{-1} = 3(1/x) = 3/x$	Three times reciprocal of x is $3(1/x)$

All the problems in the previous example contain quantities raised to the negative one power. For instance, ten raised to the negative one power written $(10)^{-1}$ is simply the reciprocal of ten. But what about ten raised to the negative two power? The following exponent rule defines quantities raised to any negative integer power.

Rule **Negative exponent rule** states that raising the base to a negative integer power is equal to the reciprocal of the base raised to the positive integer power. For any positive integer n, $(a)^{-n} = 1/(a)^n$

Example 5 Evaluate the following:

$(2)^{-3} = 1/(2)^3 = 1/8$
The reciprocal of two raised to the third power is 1/8

$3(5)^{-2} = 3(1/5)^2 = 3(1/25) = 3/25$
3 times the reciprocal of five squared is 3/25

Exponent notation with a positive integer exponent serves as shortcut notation for multiplying the same number (the base) repeatedly, while a negative integer exponent serves as notation for reciprocals, and a fractional exponent serves as notation for radicals.

Notation Raising the base to the unit fraction exponent $1/n$ indicates the n^{th} root of the base with $(a)^{1/n} = \sqrt[n]{a}$

Example 6 Evaluate the following:

$(49)^{1/2} = \sqrt{49} = 7$

Square root of 49 is 7, since 7 squared is 49.

$(27)^{1/3} = \sqrt[3]{27} = 3$

Cube root of 27 is 3, since 3 cubed is 27

$(16)^{1/4} = \sqrt[4]{16} = 2$

Fourth root of 16 is 2 since 2 raised to 4^{th} power is 16

$(25)^{-1/2} = \dfrac{1}{\sqrt{25}} = \dfrac{1}{5}$

First write as the reciprocal of square root of 25, which equals 1/5

In the previous examples, the base and exponents are given and the output of the base raised to that exponent is found. In the next examples, the output and base are given and the exponent to which that base is raised to equal a given output is found.

Example 7 Find the value of the exponent in the following:

$(5)^n = 125$

n is 3, since base 5 cubed is 125

$(2)^n = 16$

n is 4, since base 2 raised to the 4^{th} is 16

$(7)^n = 1$

n is 0, since base 7 raised to the 0 power is 1

In the next problem, the given outputs are reciprocals which result from raising the given base to negative integer exponent, so these exponents are all negative.

Example 8 Find the value of the exponent in the following:

$(3)^n = 1/81$
n is -4, since base 3 raised to the 4^{th} power is 81

$(2)^n = 1/8$
n is -3, since base 2 raised to third power is 8

$(4)^n = 1/4$
n is -1, since base 4 raised to the 1^{st} power is 4

$(6)^n = 1/36$
n is -2, since base 6 raised to 2^{nd} is 36

In the next problem, the given outputs are roots of the given base which result from raising the given base to fractional exponent, so these exponents are all fractions.

Example 9 Find the value of the exponent in the following:

$(25)^n = 5$
n is 1/2, since square root of base 25 is 5

$(8)^n = 2$
n is 1/3, since cube root of base 8 is 2

$(81)^n = 3$
n is 1/4, since 4th root of base 81 is 3

Now instead of writing an equation in exponential format with a variable as the exponent, these equations are written using logarithm notation which serves as the inverse of exponential notation. The logarithm of a value with a given base is defined as the exponent that the given base is raised to generate that value.

Definition The **logarithm** of a positive number x is the **exponent** to which the given positive base a is raised so that the result equals x.

Notation The **logarithm operation** is written as $\log_a(x) = y$ which means that base a raised to the y power equals x

$$\log_a(x) = y \quad \text{if and only if} \quad (a)^y = x$$

The following example which involves finding what power the given base is raised to equal a given value is written not in exponential format but using the more convenient logarithmic notation.

Example 10 Evaluate the following logarithms:

$\log_2(8)$	The base 2 raised to what power is equal to 8?
$\log_2(8) = 3$	The base 2 raised to the third power is 8
$\log_5(25)$	The base 5 raised to what power is equal to 25?
$\log_5(25) = 2$	The base 5 raised to the second power is 25
$\log_2(1/8)$	The base 2 raised to what power is equal to 1/8?
$\log_2(1/8) = -3$	The base 2 raised to the -3 power is 1/8
$\log_{10}(1/100)$	The base 10 raised to what power is equal to 1/100?
$\log_{10}(1/100) = -2$	The base 10 raised to the -2 power is 1/100
$\log_9(1)$	The base 9 raised to what power is equal to 1?
$\log_9(1) = 0$	The base 9 raised to the 0 power is 1
$\log_{16}(4)$	The base 16 raised to what power is equal to 4?
$\log_{16}(4) = 1/2$	The base 16 raised to the 1/2 power is 4
	In other words the square root of 16 is 4
$\log_8(2)$	The base 8 raised to what power is equal to 2?
$\log_8(2) = 1/3$	Since base 8 raised to the 1/3 power is 2
	In other words the third root of 8 is 2

Exercises 4.1

1-12. Evaluate the following without a calculator.

1.	$(4)^3$	2.	$(2)^4$	3.	$3(2)^3$
4.	$2(5)^3$	5.	$(5)^{-1}$	6.	$3(7)^{-1}$
7.	$(6)^{-2}$	8.	$(4)^{-3}$	9.	$(8)^{1/3}$
10.	$(9)^{1/2}$	11.	$(125)^{1/3}$	12.	$(36)^{-1/2}$

13-33. Evaluate the following logarithms without a calculator.

13.	$\log_3(27)$	14.	$\log_4(16)$	15.	$\log_2(8)$
16.	$\log_2(16)$	17.	$\log_5(125)$	18.	$\log_7(1)$
19.	$\log_3(1)$	20.	$\log_6(36)$	21.	$\log_2(32)$
22.	$\log_2(1/4)$	23.	$\log_3(1/27)$	24.	$\log_2(1/8)$
25.	$\log_5(1/125)$	26.	$\log_3(1/9)$	27.	$\log_9(1/81)$
28.	$\log_{16}(4)$	29.	$\log_{16}(2)$	30.	$\log_{27}(3)$
31.	$\log_9(3)$	32.	$\log_{25}(5)$	33.	$\log_{36}(6)$

Section 4.2 Common Logarithms

Below the base ten is raised to consecutive positive integer powers. Notice the following pattern that emerges. Ten raised to the first power is equal to 10, which is written as a one followed by one zero. Ten raised to the second power is equal to 100, which is written as a one followed by two zeros. Ten raised to the third power is equal to 1,000 which is written as a one followed by three zeros. Thus, ten raised to the positive n^{th} power is written as a one followed by n zeros.

$$
\begin{array}{ccccc}
(10)^1 & = & (10) & = & 10 \\
(10)^2 & = & (10)(10) & = & 100 \\
(10)^3 & = & (10)(10)(10) & = & 1{,}000 \\
\vdots & & \vdots & & \vdots \\
(10)^n & = & \underbrace{(10)(10)\cdots(10)}_{n\ tens} & = & 1\underbrace{00\cdots0}_{n\ zeros}
\end{array}
$$

Definition The **common logarithm** of a positive number is the exponent or power to which the base ten is raised so that the result equals that number. The notation for the common logarithm operation is log.

$$\log(x) = y \quad \text{if and only if} \quad (10)^y = x$$

For the remainder of this section the common logarithm is written in its abbreviated notation format as the common log or as simply the log. The common log operation serves as the inverse of the operation that raises the base ten to a power. Instead of raising base ten to a given power and finding the value, the common log starts with the given value and determines what power the base ten is raised to in order to obtain that given value. For instance, to find the common log of one hundred denoted log(100) answer the question 10 raised to what power equals one hundred? Since 10 raised to the second power denoted $(10)^2$ equals 100 it follows that the common log of 100 denoted log(100) equals 2.

Example 1 Evaluate the following common logs:

log(10,000) log(10) log(1,000,000)

log(10,000) = 4

The base ten raised to what power equals 10,000

The common log of ten thousand is four, since $(10)^4 = 10{,}000$

log(10) = 1

The base ten raised to what power equals 10

The common log of ten is one, since $(10)^1 = 10$

log(1,000,000) = 6

The base ten raised to what power equals 1,000,000

The common log of a million is six, since $(10)^6 = 1{,}000{,}000$

Below the base 10 is raised to consecutive positive integer exponents with the results written in both exponential notation and common log notation.

Exponential Notation	**Common log Notation**
$(10)^0 = 1$	$\log(1) = 0$
$(10)^1 = 10$	$\log(10) = 1$
$(10)^2 = 100$	$\log(100) = 2$
$(10)^3 = 1{,}000$	$\log(1{,}000) = 3$
$(10)^4 = 10{,}000$	$\log(10{,}000) = 4$
$(10)^5 = 100{,}000$	$\log(100{,}000) = 5$
$(10)^6 = 1{,}000{,}000$	$\log(1{,}000{,}000) = 6$
$\vdots \quad \vdots$	$\vdots \quad \vdots$
$(10)^n = 1\underbrace{00\cdots0}_{n\ zeros}$	$\log(1\underbrace{00\cdots0}_{n\ zeros}) = n$

The ability to mentally calculate the common log of place value numbers such as ten, one hundred, one thousand, ten thousand, one hundred thousand, and one million is useful in estimating the common log of other numbers. For instance, the common log of 7200 is the power to which 10 is raised in order to yield 7200. To estimate a range for the value of log(7200) notice that 7200 is between the place value numbers 1000 and 10,000. Thus, the common log of 7200 is somewhere between 3 and 4, since $10^3 = 1000$ and $10^4 = 10{,}000$. While this technique does not result in a specific value for the common log of 7200 it does provide a quick mental estimate for the range of the value as being between 3 and 4.

Example 2 Find consecutive integers that the following logs are between

log(850) log(460,000) log(7) log(9,350,000)

$2 < \log(850) < 3$

850 is between the place value numbers one hundred and one thousand.

The common log of 850 is between 2 and 3, since $10^2 = 100$ and $10^3 = 1{,}000$

$5 < \log(460{,}000) < 6$

460,000 is between the place value numbers one hundred thousand and one million.
The log of 460,000 is between 5 and 6, since $10^5 = 100{,}000$ and $10^6 = 1{,}000{,}000$

$0 < \log(7) < 1$

7 is between the place value numbers one and ten.
The common log of 7 is between 0 and 1, since $10^0 = 1$ and $10^1 = 10$.

$6 < \log(9{,}350{,}000) < 7$

9,350,000 is between the place value numbers one million and ten million.
The log of 9,350,000 is between 6 and 7, since $10^6 = 1{,}000{,}000$ and $10^7 = 10{,}000{,}000$

Negative exponents indicate reciprocals. Below the base ten is raised to consecutive negative integer powers. Notice the pattern that emerges. Ten raised to the negative one power is 0.1, which is generated by starting at one and moving the decimal point one place to the left. Ten raised to the negative two power is 0.01, which is generated by starting at one and moving the decimal point two places to the left. Ten raised to the negative three power is 0.001, which is generated by starting at one and moving the decimal point three places to the left. Thus, to find 10 raised to the negative n^{th} power, start at 1 and move the decimal place n units to the left.

$$(10)^{-1} = \frac{1}{10} = 0.1$$

$$(10)^{-2} = \frac{1}{10^2} = \frac{1}{100} = 0.01$$

$$(10)^{-3} = \frac{1}{10^3} = \frac{1}{1{,}000} = 0.001$$

$$(10)^{-n} = \frac{1}{10^n} = \frac{1}{1\underbrace{00\cdots0}_{n\ zeros}} = 0.\underbrace{00\cdots0}_{n-1\ zeros}1$$

Example 3 Evaluate the following common logs:

log(0.001) log(0.1) log(0.000001)

log(0.001) = -3

The base ten raised to what power equals 0.001

The common log of one thousandth is -3, since $(10)^{-3} = 0.001$

log(0.1) = -1

The base ten raised to what power equals 0.1

The common log of one tenth is -1, since $(10)^{-1} = 0.1$

log(0.000001) = -6

The base ten raised to what power equals 0.000001

The log of one millionth is -6, since $(10)^{-6} = 0.000001$

Below the base 10 is raised to consecutive negative integer exponents with the results written in both exponential notation and common log notation.

Exponential Notation	**Common log Notation**
$(10)^{-1} = 0.1$	$\log(0.1) = -1$
$(10)^{-2} = 0.01$	$\log(0.01) = -2$
$(10)^{-3} = 0.001$	$\log(0.001) = -3$
$\vdots \qquad \vdots$	$\vdots \qquad \vdots$
$(10)^{-n} = 0.\underbrace{00\cdots0}_{n-1\ zeros}1$	$\log(0.\underbrace{00\cdots0}_{n-1\ zeros}1) = -n$

The ability to mentally calculate the common log of place value numbers such as one tenth, one hundredth, one thousandth, one ten-thousand, one hundred-thousandth, and one millionth is useful in estimating the common log of other numbers. For instance, the common log of 0.03 is the power to which 10 is raised in order to yield 0.03 To estimate a range for the value of log(0.03) notice that 0.03 is between the place value numbers one hundredth and one tenth. Thus, the common log of 0.03 is between -2 and -1, since $10^{-2} = 0.01$ and $10^{-1} = 0.1$ While this technique does not result in a specific value for the common log of 0.03 it does provide a quick mental estimate for the range of the value as being between -2 and -1.

Example 4 Find consecutive integers that the following logs are between

$$\log(0.008) \qquad\qquad \log(0.5)$$

$-3 < \log(0.008) < -2$

0.008 is between the place value numbers one thousandth and one hundredth.
The common log of 0.008 is between -3 and -2, since $10^{-3} = 0.001$ and $10^{-2} = 0.01$

$-1 < \log(0.5) < 0$

0.5 is between the place value numbers one tenth and one.
The common log of 0.5 is between -1 and 0, since $10^{-1} = 0.1$ and $10^{0} = 1$

Prior to the widespread availability of calculators, the appendixes of textbooks contained logarithm tables that when used with various log properties determined the common logs of numbers. In this textbook the common logarithms of non-place value integers are found with a calculator using the log button. The common logs of non-place value integers are irrational numbers that in decimal form go forever without a repeating digit pattern. The calculator results for common logs are rounded to the number of decimal places specified by the problem.

Example 5 Use a calculator to evaluate the following common logarithms:

log(850) log(460,000) log(7) log(0.008) log(0)

(*round to 3 decimal places*)

log(850) ≈ 2.929 log(460,000) ≈ 5.663 log(7) ≈ 0.845

log(0.008) ≈ -2.097 log(0) undefined

In the previous example when the common logarithm of zero is entered into a calculator, the calculator does not display an output answer. The reason is that the common logarithm of zero is not a real number since ten raised to any real number power will never equal zero!

Example 6 Translate the following sentences into exponential equations. Then solve by converting into equivalent common log equations.

Ten raised to what power equals 100,000
Ten raised to what power equals 0.01
Ten raised to what power equals 430

$(10)^n = 100{,}000$	Ten raised to what power equals 100,000
$n = \log(100{,}000)$	Convert from exponential to common log notation
$n = 5$	Ten raised to the fifth power equals 100,000

$(10)^n = 0.01$	Ten raised to what power equals 0.01
$n = \log(0.01)$	Convert from exponential to common log notation
$n = -2$	Ten raised to the negative two power equals 0.01

$(10)^n = 430$	Ten raised to what power equals 430
$n = \log(430)$	Convert from exponential to common log notation.
$n \approx 2.633$	Ten raised to 2.633 power equals approximately 430

Exercises 4.2

1-9. Evaluate the following without using a calculator.

1.	log(1,000)	2.	log(100,000)	3.	log(0.0001)
4.	log(0.01)	5.	log(10)	6.	log(1)
7.	log(0.1)	8.	log(1,000,000)	9.	log(0.000001)

10. 10 raised to the 3^{rd} power is a thousand, 10 raised to the 6^{th} power is a million, 10 raised to the 9^{th} power is a billion, and 10 raised to 12^{th} power is a trillion. Rewrite the above sentence using common log notation.

11. 10 raised to the negative 3^{rd} power is one thousandth, 10 raised to the negative 6^{th} power is one millionth, 10 raised to the negative 9^{th} power is one billionth, and 10 raised to the negative 12^{th} power equals one trillionth. Rewrite the above sentence using common log notation.

12-20. Find consecutive integers that the following logs are between.
(*without the use of a calculator*)

12.	log(12,300)	13.	log(1,820)	14.	log(627,000)
15.	log(25)	16.	log(750)	17.	log(7,200,000)
18.	log(6)	19.	log(0.9)	20.	log(0.004)

21-29. Use a calculator to evaluate the following. (*round to 4 significant digits*)

21.	log(12,300)	22.	log(1,820)	23.	log(627,000)
24.	log(25)	25.	log(750)	26.	log(7,200,000)
27.	log(6)	26.	log(0.9)	27.	log(0.004)

30. Explain in sentence form why the log(-100) is not a real number.

31-34. Translate the following sentences into exponential equations. Then solve by converting into equivalent common log equations.

31. Ten raised to what power equals 1,000,000
32. Ten raised to what power equals 0.1
33. Ten raised to what power equals 17,000
34. Ten raised to what power equals 12,000,000

Section 4.3 Power property for logs

The power property of common logs is a helpful tool to solve exponential equations that model geometric sequences. To derive the power property, below the common log of 4, 8, and 16 are evaluated using a calculator and then written in terms of the common log of 2 that equals 0.301 when rounded to three decimal places. Notice the pattern established between the first and last entries on each line which is written as the common log of 2 raised to a power is equal to the product of the power times the common log of 2.

$$\begin{array}{ccccccccccc}
\log(2^2) & = & \log(4) & \approx & 0.602 & = & 2(0.301) & = & 2\cdot\log(2) \\
\log(2^3) & = & \log(8) & \approx & 0.903 & = & 3(0.301) & = & 3\cdot\log(2) \\
\log(2^4) & = & \log(16) & \approx & 1.204 & = & 4(0.301) & = & 4\cdot\log(2) \\
\vdots & & & & & & \vdots & & \vdots \\
\log(2^p) & = & & & & = & p(0.301) & = & p\cdot\log(2)
\end{array}$$

Below the common log of 9, 27, and 81 are evaluated using a calculator and then written in terms of the common log of 3 that equals 0.477 when rounded to three decimal places. Notice the pattern established between the first and last entries on each line which is written as the common log of 3 raised to a power is equal to the product of the power times the common log of 3.

$$\begin{array}{ccccccccccc}
\log(3^2) & = & \log(9) & \approx & 0.954 & = & 2(0.477) & = & 2\cdot\log(3) \\
\log(3^3) & = & \log(27) & \approx & 1.431 & = & 3(0.477) & = & 3\cdot\log(3) \\
\log(3^4) & = & \log(81) & \approx & 1.908 & = & 4(0.477) & = & 4\cdot\log(3) \\
\vdots & & & & & & \vdots & & \vdots \\
\log(3^p) & = & & & & = & p(0.477) & = & p\cdot\log(3)
\end{array}$$

Below the common log of 100, 1000, and 10000 are evaluated and then written in terms of the common log of 10 that equals 1. Notice the pattern established between the first and last entries on each line which is written as the common log of 10 raised to a power is equal to the product of the power times the common log of 10.

$$\begin{array}{ccccccccccc}
\log(10^2) & = & \log(100) & = & 2 & = & 2(1) & = & 2\cdot\log(10) \\
\log(10^3) & = & \log(1000) & = & 3 & = & 3(1) & = & 3\cdot\log(10) \\
\log(10^4) & = & \log(10{,}000) & = & 4 & = & 4(1) & = & 4\cdot\log(10) \\
\vdots & & & & & & \vdots & & \vdots \\
\log(10^p) & = & & & p & = & p(1) & = & p\cdot\log(10)
\end{array}$$

The properties for common log of two, three, and ten raised to a power are listed below. Notice how the original exponent p used becomes a factor in a product. This common log property for exponents applies not only when evaluating the common log of two, three, and ten raised to a power but when evaluating the common log of any number raised to a power.

$$\log(2^p) = p\cdot\log(2)$$

$$\log(3^p) = p\cdot\log(3)$$

$$\log(10^p) = p\cdot\log(10)$$

Property The **power property for logs** states that the common log of the quantity *a* raised to the p power is equal to the power p times the common log of *a*.

$$\log(a^p) = p\cdot\log(a)$$

Example 1 Use the power property for logs to write the expression $\log(5)^3$ without exponents. Use a calculator to verify the result.

$$\log(5^3) = 3\cdot\log(5)$$

Apply the power property for logs with 5 as the base *a* and 3 as the power p. Check using a calculator that both $\log(5^3)$ and $3\cdot\log(5)$ are equal 2.097 when rounded to three decimal places.

The power property for logs is applied when solving equations that contain a variable in the exponent such as the exponential equations that model geometric sequences. By applying the log property to both sides of these equations the variable in the exponents is dropped down and becomes part of a multiplication, which results in a linear equation that is solvable by using the principles of equality.

Example 2 Solve the following exponential equation $(1.75)^x = 5$

$(1.75)^x = 5$	The base 1.75 raised to which power yields 5
$\log(1.75)^x = \log(5)$	Take the common logarithm of both sides
$x \cdot \log(1.75) = \log(5)$	Apply power property for logs
$\dfrac{x \cdot \cancel{\log(1.75)}}{\cancel{\log(1.75)}} = \dfrac{\log(5)}{\log(1.75)}$	Divide both sides by log(1.75)
$x = \dfrac{\log(5)}{\log(1.75)}$	Exact answer
$x \approx 2.875$	Round to three decimal places
	Check with calculator that $(1.75)^{2.875} \approx 5$

Example 3 Solve the following exponential equation $2(3)^t = 15$

$2(3)^t = 15$	Isolate exponential by dividing both sides by 2
$(3)^t = 7.5$	The base 2 raised to which power yields 7.5
$\log(3)^t = \log(7.5)$	Take the common logarithm of both sides
$t \cdot \log(3) = \log(7.5)$	Apply power property for logs
$\dfrac{t \cdot \cancel{\log(3)}}{\cancel{\log(3)}} = \dfrac{\log(7.5)}{\log(3)}$	Divide both sides by log(3)
$t = \dfrac{\log(7.5)}{\log(3)}$	Exact answer
$t \approx 1.834$	Round to three decimal places
	Check with calculator that $2(3)^{1.834} \approx 15$

Now this solve technique of taking the common log of both sides of an exponential equation and applying the log property of exponents to convert the variable from an exponent to a factor is used to determine the stage at which a given geometric sequence has a given output value.

Example 4 Whitney invests \$5000 in a compounded annually account that earns 8% annual interest. Find the function that models the amount in the account. Find how many years is needed until the amount in the account is doubled.

The principal (initial value) is \$5000 and the amount increases by 8% each year. This is a geometric sequence with initial value \$5000 multiplied by 1.08 each year.

$$f(t) = 5000(1.08)^t$$

t	0	1	2	3	4	5
$f(t)$	5000.00	5400.00	5832.00	6298.56	6802.44	7346.64

To find the years required for the amount in Whitney's account to double from \$5000 to \$10,000 solve $f(t) = 10,000$ as shown below.

$$f(t) = 10,000$$

$$5000(1.08)^t = 10,000 \qquad \text{Divide both sides by 5000}$$

$$(1.08)^t = 2 \qquad \text{1.08 raised to what power equals 2}$$

$$\log(1.08)^t = \log(2) \qquad \text{Take the common log of both sides}$$

$$t \cdot \log(1.08) = \log(2) \qquad \text{Apply power property for logs}$$

$$\frac{t \cdot \cancel{\log(1.08)}}{\cancel{\log(1.08)}} = \frac{\log(2)}{\log(1.08)} \qquad \text{Divide both sides by log(1.08)}$$

$$t = \frac{\log(2)}{\log(1.08)} \qquad \text{Exact answer}$$

$$t \approx 9 \qquad \text{Round to nearest year}$$

Check by inserting 9 years into the function $f(t)$ as shown below.

$$f(9) = 5000(1.08)^9 = \$\,9995.02$$

Example 5 25,000 cells are placed in an environment in which they double in number every three hours. Find the equation that models the number of cells after t hours. Also, after how many hours are there approximately 1 million cells?

The initial value is 25,000 cells and the doubling time of 3 hours is used to generate the table below.

t hours	0	3	6	9	12	15	18
$f(t)$ cells	25,000	50,000	100,000	200,000	400,000	800,000	1,600,000

$$f(t) = 25{,}000(2)^{(t/3)}$$

The number of cells after t hours is modeled by the above exponential function since the initial value is 25,000 cells and the doubling time is 3 hours. To find when there will be 1 million cells insert 1,000,000 as the output for $f(t)$ and solve for the time t. As shown below, 16 hours later there will be more than one million cells.

$$f(t) = 1{,}000{,}000$$

$$25{,}000(2)^{(t/3)} = 1{,}000{,}000 \qquad \text{Divide both sides by a 25,000}$$

$$(2)^{(t/3)} = 40 \qquad \text{2 raised to which power yields 40?}$$

$$\log(2)^{(t/3)} = \log(40) \qquad \text{Take the common log of both sides}$$

$$\frac{t}{3}\cdot\log(2) = \log(40) \qquad \text{Apply power property of logs}$$

$$\frac{t}{3}\cdot\frac{\cancel{\log(2)}}{\cancel{\log(2)}} = \frac{\log(40)}{\log(2)} \qquad \text{Divide both sides by log(2)}$$

$$\frac{t}{3} = \frac{\log(40)}{\log(2)} \qquad \text{Multiply both sides by 3}$$

$$t = 3\cdot\frac{\log(40)}{\log(2)} \qquad \text{Use calculator to evaluate}$$

$$t \approx 16 \qquad \text{Round to nearest hour}$$

Check by inserting 16 hours into the original equation as shown below.

$$f(16) = 25{,}000(2)^{(16/3)} = 1{,}008{,}000$$

Example 6 A person drinks an espresso coffee that contains about 160 milligrams of caffeine. If caffeine in the bloodstream has a half-life of six hours, find the exponential equation that gives the amount of caffeine in the bloodstream from the coffee t hours after drinking the coffee. Also, how many hours must elapse before this person has less than 10 milligrams of caffeine in their bloodstream?

The following table is generated by starting with initial value of 160 milligrams and halving the milligrams of caffeine every 6 hours.

t hours	0	6	12	18	24	30	36
$f(t)$ mg	160	80	40	20	10	5	2.5

$$f(t) = 160(0.50)^{(t/6)}$$

The initial value is 160 milligrams and the half-life is 6 hours. The milligrams of caffeine after t hours are modeled by this exponential function. Even though the answer is available on the table, the solution can also be found by solving algebraically. To find when there will be less than 10 milligrams of caffeine insert 10 as the output for $f(t)$ and solve for the time t. As shown below, after 24 hours there will be less than 10 milligrams of caffeine in the bloodstream.

$160(0.5)^{(t/6)}$	$=$	10	Divide both sides by 160
$(0.5)^{(t/6)}$	$=$	0.0625	0.5 raised to what power equals 0.0625
$\log(0.5)^{(t/6)}$	$=$	$\log(0.0625)$	Take the common log of both sides
$\frac{t}{6}\cdot\log(0.5)$	$=$	$\log(0.0625)$	Exact answer
$\frac{t}{6}\cdot\frac{\cancel{\log(0.5)}}{\cancel{\log(0.5)}}$	$=$	$\frac{\log(0.0625)}{\log(0.5)}$	Divide each side by log(0.5)
$\frac{t}{6}$	$=$	$\frac{\log(0.0625)}{\log(0.5)}$	Multiply each side by 6
t	$=$	$6\cdot\frac{\log(0.0625)}{\log(0.5)}$	Exact answer
t	$=$	24	Round to nearest hour

Exercises 4.3

1-4. Use the power property for logs to write the following expressions without exponents. Use a calculator to verify the result.

1. $\log(2^8)$ 2. $\log(4^3)$ 3. $\log(8^6)$ 4. $\log(5^7)$

5-8. Solve the following exponential equations.

5. $8(2)^t = 200$ 6. $5(1.40)^t = 62$

7. $100(2)^{(t/7)} = 20{,}000$ 8. $200(0.5)^{(t/5)} = 4$

9. Debbie invests \$10,000 in a compounded annually account that earn 5% annual interest.

9A. Find the equation that models the balance of this account after *t* years.
9B. How many years later is the balance in Debbie's account \$15,000?
9C. How many years later before the original amount is doubled?

10. Jamie invests \$3000 in a compounded annually account that earn 4% annual interest.

10A. Find the equation that models the balance of this account after *t* years.
10B. How many years later is the balance in Jill's account \$4000?
10C. How many years later before the original amount is doubled?

11. Jayson invests \$25,000 in a compounded annually account that earn 3.5% annual interest.

11A. Find the equation that models the balance of this account after *t* years.
11B. How many years later is the balance in Jayson's account \$30,000?
11C. How many years later before the original amount is doubled?

12. A small town has a population of 50,000 residents in 2016 and the population is projected to increase by 3% each year thereafter.

12A. Find the function $f(t)$ that models the town's population *t* years later.
12B. According to this model when will the population hit 60,000 residents?
12C. How many years later before the original population is doubled?

13. The current world population in 2015 is 7.4 billion and is currently growing at the rate of 1.13% per year.

13A. Find the function $f(t)$ that models the world population t years later.

13B. According to this model when will the population hit 9 billion?

13C. How many years later before the original population is doubled?

14. 10,000 cells are placed in an environment in which they increase in number by 40% each hour.

14A. Find the equation that models the number of cells t hours later.

14B. When are there 1 million cells?

15. 50,000 cells are placed in an environment in which they double in number every four hours.

15A. Find the equation that models the number of cells t hours later.

15B. When are there 1 million cells?

16. 100,000 cells are placed in an environment in which they double in number every six hours.

16A. Find the equation that models the number of cells t hours later.

16B. When are there 1 million cells?

17. A given drug has a half-life of eight hours in a patient's bloodstream and a patient is injected with 120 milligrams of this medication.

17A. Find the equation that models the milligrams of this medication in the patient's bloodstream after t hours.

17B. How many hours must elapse before this patient has less than 10 milligrams of this medication in their bloodstream?

18. A given drug has a half-life of six hours in a patient's bloodstream and a patient is injected with 100 milligrams of this medication.

18A. Find the equation that models the milligrams of this medication in the patient's bloodstream after t hours.

18B. How many hours must elapse before this patient has less than 20 milligrams of this medication in their bloodstream?