Chapter 3: Exponential Functions and Exponential Regression

Section 3.1 Geometric Sequences

In chapter two, arithmetic sequences are described as patterns generated by adding a constant number to the previous number on the list. In this section a geometric sequence is introduced. These patterns occur in numerous applications including compound interest accounts and growth models.

Definition An **arithmetic sequence** is a numerical pattern that is generated by adding a constant to the previous number on the list.

Definition A **geometric sequence** is a numerical pattern that is generated by multiplying a constant times the previous number on the list.

To generate arithmetic and geometric sequences the same operation is applied to the previous number on the list, but for arithmetic sequences the operation applied is addition whereas for geometric sequences the operation applied is multiplication.

Example 1 Are the following patterns arithmetic or geometric.

2, 6, 18, 54, 162, …
This is a geometric sequence with initial value of 2 that is generated by multiplying 3 times the previous number on the list.

2, 5, 8, 11, 14, …
This is an arithmetic sequence with initial value of 2 that is generated by adding 3 to previous number on the list by three.

32, 16, 8, 4, 2, …
This is a geometric sequence with initial value of 32 that is generated by multiplying ½ times the previous number on the list. Note multiplication by one-half is same as division by two.

Example 2 List the first five values in the geometric sequence that has an initial value of 16 and is multiplied by 1.50 at each stage.

The initial value of this sequence is 16. To generate this sequence multiply by 1.50 at each stage: 16(1.50) = 24, 24(1.50) = 36, 36(1.50) = 54, and 54(1.50) = 81 The resulting geometric sequence is 16, 24, 36, 54, 81, …

x	0	1	2	3	4
y	16	24	36	54	81

All arithmetic sequences have a constant change (increase or decrease) between consecutive values in the pattern. Is there a similar constant relationship between consecutive values in a geometric sequence? Consider the geometric sequence 16, 24, 36, 54, 81, … generated by multiplying the previous number by 1.5, since this is not an arithmetic pattern the increase per stage is not constant as shown below.

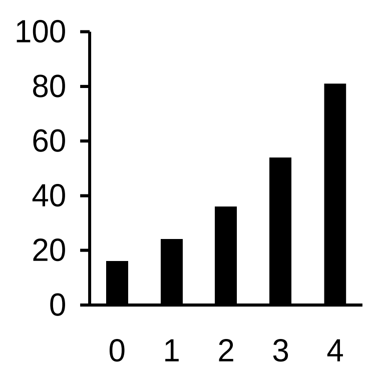
Consecutive numbers	Increase per stage
16 and 24	24 – 16 = 8
24 and 36	36 – 24 = 12
36 and 54	54 – 36 = 18

Another measurement is the percentage change (increase or decrease) at each stage of the sequence. To find the percentage change between two consecutive values in a sequence divide the increase or decrease by the first of the two value. The percentage increase is calculated for consecutive values in the geometric sequence 16, 24, 36, 54, 81, … in the table below. This geometric sequence has a constant percentage increase of 50% at each stage. This geometric pattern 16, 24, 36, 54, 81, … is generated either by starting at initial value of 16 and multiplying by 1.50 at each stage or by starting at the initial value of 16 and increasing by 50% per stage. Now returning to the earlier question is there a relationship between consecutive values in a geometric sequence?

Consecutive numbers	Increase per stage	% increase per stage
16 and 24	24 – 16 = 8	8/16 = 0.50 = 50%
24 and 36	36 – 24 = 12	12/24 = 0.50 = 50%
36 and 54	54 – 36 = 18	18/36 = 0.50 = 50%

Consecutive values in a geometric sequence do not have a constant rate of change, but consecutive values in a geometric sequence do have a constant percentage change. Below the geometric sequence 16, 24, 36, 54, 81, … is displayed as an input/output table and drawn as a bar graph. At each successive stage the bar is 50% higher than the previous bar, which represents the constant percentage increase of 50% per stage.

x	0	1	2	3	4
y	16	24	36	54	81



Arithmetic sequences are generated by the linear function $y = ax + b$. Is there a similar function that generates geometric patterns? Consider the geometric sequence 16, 24, 36, 54, 81… with an initial value of 16 that is generated by multiplying by 1.50 at each stage and which increases by 50% at each stage. This geometric sequence is displayed in the following table on the left. To find y, the value of this geometric pattern at the x^{th} stage, multiply the initial value 16 times the product that contains the number 1.50 listed x times. Exponential form serves as a shortcut notation for repeated multiplication of the same number. The product that contains the number 1.50 listed x times is written in exponential notation as $(1.50)^x$. The function $f(x) = 16(1.50)^x$ quickly finds the value of this geometric pattern at any stage. Since this function involves the variable x, the stage of the sequence, as an exponent it is called an exponential function. The function for the geometric sequence with initial value b multiplied by a at each stage is derived in the table on the right. To find y, the value of this geometric pattern at the x^{th} stage, multiply the initial value b times the product of a listed x times. This product of b listed x times is written in exponential notation as $(a)^x$. The exponential function $f(x) = b(a)^x$ models a geometric sequence with initial value b multiplied by a at each stage.

Stage	Write using exponential notation
0	16
1	$16(1.50) = 16(1.50)^1$
2	$16(1.50)(1.50) = 16(1.50)^2$
3	$16(1.50)(1.50)(1.50) = 16(1.50)^3$
x	$16\underbrace{(1.50)(1.50)\cdots(1.50)}_{x \text{ groups of } 1.50} = 16(1.50)^x$

Stage	Write using exponential notation
0	b
1	$b(a) = b(a)^1$
2	$(b)(a)(a) = b(a)^2$
3	$(b)(a)(a)(a) = b(a)^3$
x	$(b)\underbrace{(a)(a).......(a)}_{a \text{ appears } x \text{ times}} = b(a)^x$

A geometric sequence can be generated in two different ways either starting at the initial value b and multiplying by the constant number a at each stage or starting at the initial value b and increasing (or decreasing) by a constant percentage at each stage. For an increasing geometric sequence what is the connection between the constant multiplied by at each stage and the constant percentage increase per stage? To find this connection, consider the following cases that occur when two positive numbers are multiplied together.

(1) If the second number is one, then the first number remains unchanged.

(2) If the second number is greater than one, then the product is larger than the first number.

Multiplying a positive number by 1 leaves it unchanged but multiplying a positive number by a number greater than 1 makes it larger. The question is how much larger does the first number *b* get after it is multiplied by a second number *a* which is greater than 1.00? Consider the product of 10 times 1.40, since 1.40 is greater than 1.00 the product 14 is larger than the first number 10, but how much larger? The product of 10 and 1.40 equals 14, which is 40% larger than the original number 10. Notice 1.40 – 1.00 also equals .40 which represent the 40% increase when multiplying by 1.40 This relationship between the constant number *a* multiplied at each stage and the constant percentage increase per stage is illustrated in the table below. For any increasing geometric sequence the difference between the number multiplied at each stage and 1.00 is equal to the constant percentage increase per stage. In order words, the percentage increase per stage is how much the number multiplied by at each stage is greater than 1.00

b	***a***	**Product (*b*)(*a*)**	**Actual Increase**	**% Increase**	***a* – 1.00 = % increase**
10	1.40	14	4	4/10 = 40%	1.40 – 1.00 = .40 = 40%
20	1.80	36	16	16/20 = 80%	1.80 – 1.00 = .80 = 80%
40	1.05	42	2	2/40 = 5%	1.05 – 1.00 = .05 = 5%
60	3	180	120	120/60 = 200%	3.00 – 1.00 = 2 = 200%

Amount of increase 4

10 **Multiply by 1.40** **14**

4/10 = 40% increase

Amount of increase 16

20 **Multiply by 1.80** **36**

16/20 = 80% increase

Amount of increase 2

40 **Multiply by 1.05** **42**

2/40 = 5% increase

Amount of increase 120

60 **Multiply by 3** **180**

120/60 = 200% increase

Definition The exponential function $f(x) = b(a)^x$ with $a > 1.00$ generates an increasing geometric sequence with *b* the initial value and *a* the constant number that is multiplied times the previous number on the list. The constant percentage increase is the difference between the number multiplied by at each stage *a* and 1.00

Example 3 Find the exponential function that generates the geometric sequence with an initial value of 50 that is multiplied by 3 at each stage.

The exponential function $f(x) = 50(3)^x$ generates the geometric sequence with initial value 50 that is multiplied by 3 at each stage. Since 3 the number multiplied by at each stage is 2 larger than 1.00, this geometric sequence increases by 200% per stage.

x	0	1	2	3	4
y	50	150	450	1350	4050

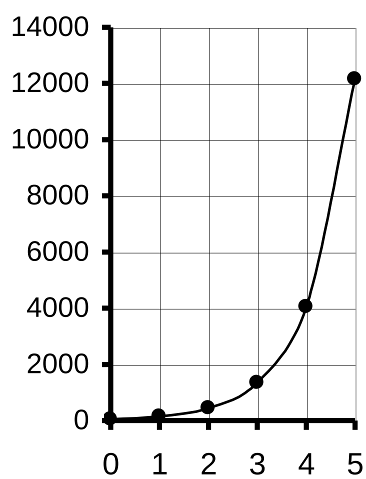
Example 4 Find the exponential function that generates the geometric sequence with initial value 20 that increases by 10% at each stage.

The exponential function $f(x) = 20(1.10)^x$ increases by 10% per stage since 1.10 is 0.10 larger than 1.00 and 100% + 10% = 110%. This geometric sequence has an initial value of 20 and is multiplied by 1.10 at each stage.

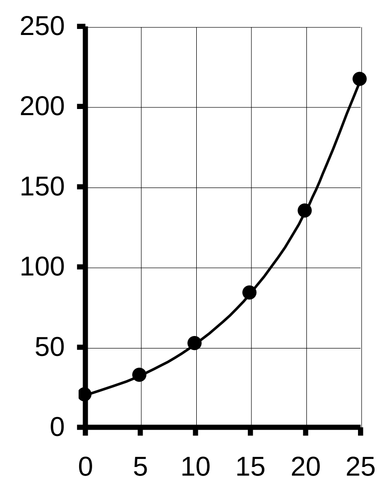
x	0	1	2	3	4
$f(x)$	20	22	24.2	26.6	29.3

The graph of an exponential function $f(x) = b(a)^x$ with $b > 1$ generates an increasing geometric sequence which results in a **J** shaped growth curve. The graphs of $y = 50(3)^x$ and $y = 20(1.10)^x$ from examples 3 and 4 are shown below. With the exponential function $f(x) = 50(3)^x$ that increases at 200% per stage the **J** shaped growth curve occurs quickly in the graph while with the function $f(x) = 20(1.10)^x$ that increases at only 10% per stage it takes about 25 stages to form the **J** shaped growth curve.

$y = 50(3)^x$



$y = 20(1.10)^x$



For an increasing geometric sequence the constant percentage increase is the difference between 1 and the number multiplied by at each stage *a*. But, for a decreasing geometric sequence what is the connection between the constant multiplied by at each stage and the constant percentage decrease per stage. To discover this connection, consider the following cases that occur when two positive numbers are multiplied together.

(1) If the second number is one, then the first number remains unchanged.

(2) If the second number is smaller than one, then the product is smaller than the first number.

Multiplying a positive number by 1 leaves it unchanged but multiplying a positive number by a positive number less than 1 makes it smaller. The question is how much smaller does the first number *b* get after it is multiplied by a second positive number *a* which is less than 1.00? Consider the product of 10 times .40, since .40 is less than 1.00 the product 4 is smaller than the first number 10, but how much smaller? The product of 10 and .40 equals 4, which is 60% less than the original number 10. Notice that 1.00 – .40 also equals 0.60 which represent the 60% decrease when multiplying by .40 This relationship between the constant number *a* multiplied at each stage and the constant percentage decrease per stage is illustrated in the table below. For any decreasing geometric sequence the difference between 1.00 and *a* the number multiplied at each stage and is equal to the constant percentage decrease per stage. In order words, the percentage decrease is how much the number multiplied by at each stage *a* is less than 1.00

b	a	$(b)(a)$ Product	Actual Decrease	% Decrease	$1.00 - a =$ % decrease
10	0.40	4	6	6/10 = 60%	1.00 – .40 = .60 = 60%
60	0.95	57	3	3/60 = 5%	1.00 – .95 = .05 = –5%

Amount of decrease 6

10 **Multiply by .40** **4**

6/10 = 60% decrease

Amount of decrease 3

60 **Multiply by .95** **57**

3/60 = 5% decrease

Definition The exponential function $f(x) = b(a)^x$ with $0 < a < 1.00$ generates a decreasing geometric sequence with b the initial value and a the constant number that is multiplied times the previous number on the list. The constant percentage decrease is the difference between 1.00 and a.

Example 5 Find the function that generates the geometric sequence with an initial value of 200 that is multiplied by .75 at each stage.

The exponential function $f(x) = 200(.75)^x$ generates the geometric sequence with initial value 200 that is multiplied by .75 at each stage. Since .75 the number multiplied by at each stage is .25 less than 1.00 this geometric sequence decreases by 25% per stage. Another way to look at this since 100% – 75% = 25%, by multiplying by 0.75 at each stage 75% remains and 25% is lost each stage.

x	0	1	2	3	4
y	200	150	112.5	84.4	63.3

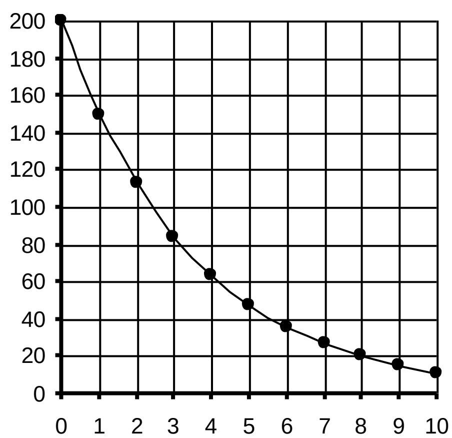
Example 6 Find the function that generates the geometric sequence with initial value of 80 that decreases by 40% at each stage.

The exponential function $f(x) = 80(.60)^x$ decreases by 40% at each stage since .60 is .40 less than 1.00 This geometric sequence has an initial value of 80 and is multiplied by .60 at each stage. Another way to look at this, since 100% – 60% = 40%, by multiplying by .60 at each stage 60% remains and 40% is lost each stage.

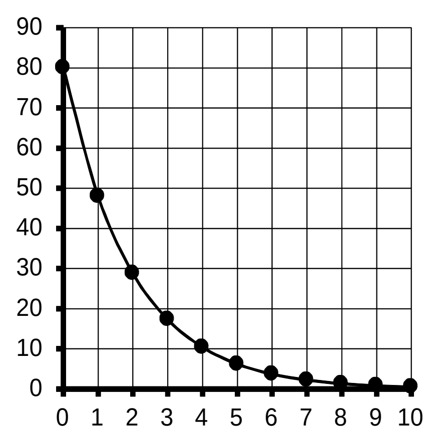
x	0	1	2	3	4
$f(x)$	80	48	28.8	17.3	10.4

The graph of an exponential function $f(x) = b(a)^x$ with $0 < a < 1$ generates a decreasing geometric sequence which results in a decay curve that resembles the mirror image of the letter J. The graphs of the decreasing exponential functions from examples 5 and 6, $y = 200(0.75)^x$ and $y = 80(0.60)^x$ are shown below.

$y = 200(0.75)^x$



$y = 80(0.60)^x$



Example 7 Determine if the following sequences are arithmetic or geometric and find the formula that generates each sequence.

20, 16, 12, 8, … 400, 320, 256, 204.8, …

20, 16, 12, 8, …

This is an arithmetic sequence with initial value 20 that decreases by 4 each stage. The linear function $f(x) = -4x + 20$ generates this arithmetic sequence.

400, 320, 256, 204.8, …

This is not an arithmetic sequence since it does not decrease by a constant amount at each stage. To determine if this is a geometric sequence find the quotient obtained from dividing consecutive numbers, with 320/400 = 256/320 = 204.8/256 = 0.80. Since this quotient is constant, this is a geometric sequence with initial value 400 which is multiplied by 0.80 at each stage.

The exponential function $f(x) = 400(0.80)^x$ generates this geometric sequence which decreases by 20% each stage. Since 100% – 80% = 20%, by multiplying by 0.80 at each stage 80% remains and 20% is lost each stage.

Example 8 Find the geometric sequence that has an initial value of 40 and the value at the first stage is 50.

Start by creating a table using the two given data points (0 , 40) and (1 , 50) Multiplying at each stage by a constant number generates geometric patterns. To find this constant number given two consecutive values divide the first value by the initial value which in this case results in $a = 50/40 = 1.25$. The exponential function $f(x) = 40(1.25)^x$ generates this geometric pattern which increases by 25% per stage since 100% + 25% = 125% which equals 1.25

x	**0**	**1**	2	3	4
y	**40**	**50**	62.5	78.1	97.7

Exercises 3.1

1-8. Determine if the following sequence are arithmetic, geometric, or neither and circle the appropriate type. **If** the patterns are arithmetic or geometric find the function that generates the sequence.

1. 8, 5, 2, -1, -4, … y = ______________
   arithmetic, geometric, or neither

2. 400, 200, 100, 50, 25, … y = ______________
   arithmetic, geometric, or neither

3. 2, 10, 50, 250, 1250, … $f(x)$ = ______________
   arithmetic, geometric, or neither

4. 40, 60, 80, 100, 120, … $f(x)$ = ______________
   arithmetic, geometric, or neither

5. 2, 4, 8, 14, 22, 32, 44, … y = ______________
   arithmetic, geometric, or neither

6. 40, 60, 90, 135, 202.5, … y = ______________
   arithmetic, geometric, or neither

7. 100, 120, 144, 172.8, 207.36, … $f(x)$ = ______________
   arithmetic, geometric, or neither

8. 200, 120, 72, 43.2, 25.92, … $f(x)$ = ______________
   arithmetic, geometric, or neither

9-12. Fill in the table with values from following geometric sequences. Find the percentage increase or decrease per stage.

9. $y = 3(2)^x$ _____ %

x	0	1	2	3	4
y					

10. $f(x) = 100(1.60)^x$ _____ %

x	0	1	2	3	4
y					

11. $y = 90\left(0.50\right)^x$ _____ %

x	0	1	2	3	4
y					

12. $f(x) = 100(0.80)^x$ _____ %

x	0	1	2	3	4
y					

13-18 Create a table of values for the following sequences and find the function that generates the pattern.

13. Arithmetic sequence with initial value 7 that increases by 3 units per stage. $y =$ ________________

x	0	1	2	3	4
y	7				

14. Geometric sequence with initial value 10 multiplied by 1.2 at each stage. $f(x) =$ ________________

x	0	1	2	3	4
y	10				

15. Arithmetic sequence with initial value 20 and 16 the value at the first stage. $f(x) =$ ________________

x	0	1	2	3	4
y	20	16			

16. Geometric sequence with initial value 20 and 16 the value at the first stage. $y =$ ________________

x	0	1	2	3	4
y	20	16			

17. Geometric sequence with initial value 80 that increases by 20% at each stage. $f(x) =$ ________________

x	0	1	2	3	4
y	80				

18. Geometric sequence with initial value 100 that decreases by 25% per stage. $y =$ ________________

x	0	1	2	3	4
y	100				

19A. A job pays \$15.00 per hour for an initial training day and you are required to work eight hours each day. Each day for the first month you receive a \$.20 raise in your hourly pay. Fill in the table below that gives the daily salary earned for each day of work. For instance, on the initial day the daily salary earned is 8(\$15.00) = \$120.00 and on the next day your daily salary is 8(\$15.20)

x days	0	1	2	3	4	5
y \$	120.00					

19B. Does the daily salary earned form an arithmetic or geometric sequence? Find the function $f(x)$ that generates this daily salary sequence.

$f(x)$ = ________________

19C. Find your daily salary after working there for a month assuming twenty workdays in a month? *note 20th day of work is represented by x = 19*

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20A. A job pays \$.01 per hour for an initial training day and you work eight hours a day. Each day for your hourly pay rate is doubled. Fill in the table below that gives the daily salary earned for each day of work. For instance, on the initial day the daily salary earned is 8(\$0.01) = \$0.08 and on the next day the daily salary earned is 8(\$.02) and so on.

x days	0	1	2	3	4	5
y \$	0.08					

20B. Does the daily salary earned form an arithmetic or geometric sequence? Find the function $f(x)$ that generates this daily salary sequence.

$f(x)$ = ________________

20C. Find your daily salary after working there for a month assuming twenty workdays in a month? *note 20th day of work is represented by x = 19*

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Section 3.2 Review of Scientific Notation and Exponent Rules

Scientific notation expresses numbers uniquely in a compact format that contains ten raised to an integer power. Scientific notation is especially useful in representing very large quantities such as the distance in miles between two planets or the number of carbon atoms in a gram of carbon and very small quantities such as the size of a virus or the toxicity levels of certain chemicals. Below the base ten is raised to consecutive positive integer powers resulting in consecutive place value numbers.

$$(10)^1 = (10) = 10$$
$$(10)^2 = (10)(10) = 100$$
$$(10)^3 = (10)(10)(10) = 1{,}000$$
$$(10)^4 = (10)(10)(10)(10) = 10{,}000$$
$$(10)^5 = (10)(10)(10)(10)(10) = 100{,}000$$
$$(10)^6 = (10)(10)(10)(10)(10)(10) = 1{,}000{,}000$$

Notice the following pattern that emerges. Ten raised to the first power is equal to 10, which is written as a one followed by one zero. Ten raised to the second power is equal to 100, which is written as a one followed by two zeros. Ten raised to the third power is equal to 1,000 which is written as a one followed by three zeros. Ten raised to the fourth power is equal to 10,000 which is written as a one followed by four zeros. Thus, ten raised to the positive n^{th} power is written as a one followed by n zeros. Alternatively, to evaluate $(10)^n$ start at the number 1 and move the decimal place n places to the right.

Example 1 Simplify $(10)^9$ and $3(10)^5$

$(10)^9 = 1{,}000{,}000{,}000$

Write ten raised to the ninth power as a one followed by nine zeros which is equal to one billion.

$3(10)^5 = 3(100{,}000) = 300{,}000$

First write ten raised to the fifth power as a one followed by five zeros which is equal to one hundred thousand, then multiply three times one hundred thousand.

Negative exponents indicate reciprocals. Below the base ten is raised to consecutive negative integer powers which results in consecutive place value decimals one tenth, one hundredth, one thousandths, …

$$(10)^{-1} = 1/(10) = 0.1$$

$$(10)^{-2} = 1/(10)^2 = 1/100 = 0.01$$

$$(10)^{-3} = 1/(10)^3 = 1/1000 = 0.001$$

$$(10)^{-4} = 1/(10)^4 = 1/10{,}000 = 0.0001$$

$$(10)^{-5} = 1/(10)^5 = 1/100{,}000 = 0.00001$$

Notice the pattern that emerges. Ten raised to the negative one power is 0.1, which is generated by starting at one and moving the decimal point one place to the left. Ten raised to the negative two power is 0.01, which is generated by starting at one and moving the decimal point two places to the left. Ten raised to the negative three power is 0.001, which is generated by starting at one and moving the decimal point three places to the left. Ten raised to the negative four power is 0.0001, which is generated by starting at one and moving the decimal point four places to the left. Thus, to find ten raised to the negative n^{th} power, start at one and move the decimal place n units to the left.

Example 2 Simplify $(10)^{-6}$ and $7(10)^{-3}$

$(10)^{-6} = 0.000001$

To evaluate ten raised to the negative six power start at one and move the decimal places six places to the left, which is equal to one millionth.

$7(10)^{-3} = 7(0.001) = 0.007$

To evaluate ten raised to the negative three power start at one and move the decimal places three places to the left, which is equal to one thousandth. Then multiply seven times 0.001

Definition **<u>Scientific notation format</u>** $A(10)^n$ or $A \times 10^n$ describes a positive number uniquely with exponent n an integer and the coefficient A greater than or equal to one but less than ten.

The goal of scientific notation is to uniquely write numbers in a format containing ten raised to an integer power. For example, \$2000 is equivalent to two \$1000 bills or twenty \$100 bills or two hundred \$10 bills. Similarly the number 2000 can be written with ten raised to a power as either $2(10)^3$, $20(10)^2$, or $200(10)^1$. But, the number 2000 is written in scientific notation uniquely as $2(10)^3$ since the coefficient two is between one and ten and the power three is an integer. Even though $20(10)^2$ and $200(10)^1$ both equal 2,000 neither is written in scientific notation format since their respective coefficients twenty and two hundred are not between one and ten.

Example 3 Which of the following are written in scientific notation format?

$12(10)^5$ $\qquad$ $3.23 \times (10)^{-4}$ $\qquad$ $7.5(8)^6$

$12(10)^5$ is **not** written in scientific notation format, since the coefficient twelve is not between one and ten.

$3.23 \times (10)^{-4}$ is written in scientific notation format, since the coefficient 3.23 is between one and ten and the exponent negative four is an integer.

$7.5(8)^6$ is **not** written in scientific notation, since the base is eight and not ten.

Scientific notations user-friendly nature is based on the simplicity of multiplying by place value numbers. This is illustrated in the following chart in which the number 2.45 is multiplied times ten raised to various integer powers. Notice that multiplying 2.45 times ten raised to an integer power simply moves the decimal place so many places to either the right or left depending on the sign and the value of the power.

$2.45(10)^1 = 2.45(10) = 24.5$	$2.45(10)^{-1} = 2.45(0.1) = 0.245$
$2.45(10)^2 = 2.45(100) = 245$	$2.45(10)^{-2} = 2.45(0.01) = 0.0245$
$2.45(10)^3 = 2.45(1{,}000) = 2450$	$2.45(10)^{-3} = 2.45(0.001) = 0.00245$
$2.45(10)^4 = 2.45(10{,}000) = 24{,}500$	$2.45(10)^{-4} = 2.45(0.0001) = 0.000245$

Rule $\quad$ n is positive integer. To multiply a positive number times $(10)^n$ the resulting product, which is larger than the original number, is found by moving the decimal point of the number n units to the right.

Rule n is a positive integer. To multiply a positive number times $(10)^{-n}$ the resulting product, which is smaller than the original number, is found by moving the decimal point of the number n units to the left.

The multiplication rules quickly transfers numbers written in scientific notation format into standard form. For positive exponents n the number is larger than 10 with the decimal place moved n places to the right and for negative exponents $-n$ the decimal is smaller than 1 with the decimal place moved n places to the left.

Example 4 Write the following numbers in standard form.

$2.7(10)^9 \quad = \quad 2,\underline{\mathbf{700,000,000}} \quad = \quad 2,700,000,000$

Start at 2.7 and since multiplying by $(10)^9$ makes the product larger move the decimal point nine places to the right.

$3.12 \times (10)^{-5} \quad = \quad 0.\underline{\mathbf{00003}}12 \quad = \quad 0.0000312$

Start at 3.12 and since multiplying by $(10)^{-5}$ makes the product smaller move the decimal point five places to the left.

$4.91(10)^4 \quad = \quad 4\underline{\mathbf{9,100}} \quad = \quad 49,100$

Start at 4.91 and since multiplying by $(10)^4$ makes the product larger move the decimal point four places to the right.

The multiplication rules also quickly transfer numbers written in standard form into scientific notation form. First find the coefficient between 1 and 10 and indicate it with an arrow as shown below. If the number is larger than 10 the exponent is positive while if the decimal is smaller than 1 then the exponent is negative.

Example 5 Write following numbers in scientific notation form.

$123,000 \quad = \quad 1\underline{\mathbf{23,000}} \quad = \quad 1.23(10)^5$

Since 123,000 is larger than 10 the exponent is positive. Insert an arrow to indicate a coefficient 1.23 that is between 1 and 10. Multiply 1.23 by $(10)^5$ to move the decimal place five places to the right that results in the original number.

$38,000,000 \quad = \quad 3\underline{\mathbf{8,000,000}} \quad = \quad 3.8(10)^7$

Since 38 million is larger than 10 the exponent is positive. Insert an arrow to indicate a coefficient 3.8 that is between 1 and 10. Multiply 3.8 by $(10)^7$ to move the decimal place seven places to the right that results in the original number.

$0.00000521 \quad = \quad 0.\underline{\mathbf{000005}}21 \quad = \quad 5.21(10)^{-6}$

Since 0.00000521 is smaller than 1 the exponent is negative. Insert an arrow to indicate a coefficient 5.21 that is between 1 and 10. Multiply 5.21 by $(10)^{-6}$ to move the decimal place six places to the left that results in the original number.
The multiplication and division rule for exponents are listed below. These rules are used to multiply and divide numbers written in scientific notation form in which both have the common base 10. To multiply numbers written in scientific notation format multiply the coefficients and raise 10 to the sum of their powers, while to divide numbers written in scientific notation divide their coefficients and raise 10 to the difference of their powers.

Multiplication rule for exponents

$$a^n a^m \quad = \quad a^{n+m}$$

To multiply monomials with the same base add their exponents and raise the common base to that power

Division rule for exponents

$$\frac{a^n}{a^m} \quad = \quad a^{n-m}$$

To divide monomials with the same base subtract their exponents and raise the common base to that power

Example 6 Use the exponent rules simplify the following.

$$\frac{84,000}{2,000,000,000} \quad = \quad \frac{8.4(10)^4}{2(10)^9} \quad = \quad 4.2(10)^{4-9} \quad = \quad 4.2(10)^{-5}$$

First write each number in scientific notation form. Divide the coefficients 8.4 and 2 and then apply the division rule for exponents to subtract the exponents 4 and 9.

$$(1,200,000)(0.0003) \quad = \quad 1.2(10)^6 \times 3(10)^{-4} \quad = \quad (1.2)(3)(10)^{6+-4} \quad = \quad 3.6(10)^2$$

First write each number in scientific notation form. Multiply the coefficients 1.2 and 3 and then apply the multiplication rule for exponents to add the exponents 6 and -4.

$$\frac{7,000,000}{0.002} \quad = \quad \frac{7(10)^6}{2(10)^{-3}} \quad = \quad 3.5(10)^{6--3} \quad = \quad 3.5(10)^9$$

First write each number in scientific notation form. Divide the coefficients 7 and 2 and then apply the division rule for exponents to subtract the exponents 6 and -3.

The multiplication and division of numbers written in scientific notation format can result with coefficients that are not between one and ten. In these cases the resulting coefficient is converted into scientific notation format and then the multiplication rule for exponents is applied to add the exponents of the base ten values as shown in the following examples.

Example 7 Write $720(10)^{14}$ and $0.23(10)^{-5}$ in scientific notation format.

$$720(10)^{14} \;=\; 7\underline{\mathbf{20}}(10)^{14} \;=\; 7.2(10)^{2}(10)^{14} \;=\; 7.2(10)^{2+14} \;=\; 7.2(10)^{16}$$

Since the original coefficient 720 is greater than 10 insert an arrow to indicate a coefficient 7.2 that is between 1 and 10 and write 720 in scientific notation as $7.2(10)^2$. Now apply the multiplication rule for exponents to add the exponents 2 and 14.

$$0.23(10)^{-5} \;=\; 0.\underline{\mathbf{2}}3(10)^{-5} \;=\; 2.3(10)^{-1}(10)^{-5} \;=\; 2.3(10)^{-1\,+\,-5} \;=\; 2.3(10)^{-6}$$

Since the original coefficient 0.23 is less than 1 insert an arrow to indicate a coefficient 2.3 that is between 1 and 10 and write 0.23 in scientific notation as $2.3(10)^{-1}$. Now apply the multiplication rule for exponents to add the exponents -1 and -5.

Example 8 Use the exponent rules simplify the following.

$$8(10)^{9} \times 4(10)^{-6} \;=\; (8)(4)(10)^{9\,+\,-6} \;=\; 32(10)^{3} \;=\; 3.2(10)^{1}(10)^{3} \;=\; 3.2(10)^{4}$$

Multiply the coefficients 8 and 4 and apply the multiplication rule for exponents to add the exponents 9 and -6. Since the resulting coefficient 32 larger than 10, write 32 in scientific notation as $3.2(10)^1$ and use the multiplication rule for exponents to add the exponents 1 and 3.

$$\frac{2(10)^{19}}{5(10)^{6}} \;=\; \frac{2}{5}(10)^{19-6} \;=\; 0.4(10)^{13} \;=\; 4(10)^{-1}(10)^{13} \;=\; 4(10)^{-1\,+\,13} \;=\; 4(10)^{12}$$

Divide the coefficients 2 and 5 and apply the division rule for exponents to subtract the exponents 19 and 6. Since the resulting coefficient 0.4 is less than 1, write 0.4 in scientific notation as $4(10)^{-1}$ and use the multiplication rule for exponents to add the exponents -1 and 13.

Exercises 3.2

1-9. Write the following numbers in standard form.

1.	$5.2(10)^4$	2.	$2(10)^8$	3.	$9.2(10)^{-5}$
4.	$4.31(10)^{-7}$	5.	$4.6(10)^5$	6.	$8.12(10)^{-6}$
7.	$7.96(10)^{-4}$	8.	$9.257(10)^{12}$	9.	$3(11)^2$

10-21. Write the following numbers in scientific notation format.

10.	230,000,000	11.	3,421,000,000	12.	0.00012
13.	0.0000123	14.	27,000	15.	0.00000571
16.	0.00004	17.	92,000,000	18.	$75(10)^8$
19.	$850(10)^{-7}$	20.	$0.241(10)^{10}$	21.	$0.12(10)^{-4}$

22. Which of the following numbers are written in scientific notation format?

$7.2(10)^4$ $\quad 15.7(10)^9$ $\quad 2(10)^{-6}$ $\quad 1.43(10)^{12}$ $\quad 1.2(20)^4$

23. A millisecond is a thousandth of a second and a microsecond is a millionth of a second. Write a millisecond and microsecond in scientific notation.

24. In chemistry calculations, a mole is used to indicate $6.022(10)^{23}$ items. Write a sentence which describes how a mole is written in standard form.

25-32. Use the exponent rules to perform the indicated operations on the following expressions. Write the final answers in scientific notation format.

25.	$1.2(10)^4 \times 5(10)^3$	26.	$3(10)^5 \times 2.1(10)^7$
27.	$1.5(10)^7 \times 6(10)^{-5}$	28.	$5.1(10)^5 \times 3(10)^4$
29.	$4.2(10)^{-13} \times 5(10)^8$	30.	$\dfrac{6(10)^{11}}{1.5(10)^5}$

31. $\dfrac{6(10)^5}{1.5(10)^{11}}$

32. $\dfrac{8(10)^7}{4(10)^{-3}}$

33-40. Convert all the following numbers into scientific notation and then use the exponent rules to perform the indicated operations on the following expressions. Write final answers in scientific notation format.

33. (23,000,000)(3,000)

34. (4,200,000)(0.000002)

35. (5,200,000)(6,000,000)

36. (23,000)(0.00000007)

37. $\dfrac{9{,}000}{3{,}000{,}000}$

38. $\dfrac{6{,}200{,}000}{31{,}000}$

39. $\dfrac{6{,}200{,}000}{0.000031}$

40. $\dfrac{0.0045}{0.000005}$

41. The mass of one neutron is $1.675(10)^{-27}$ kilograms. Find the mass of a trillion neutrons?

42. The speed of light is approximately 186,000 miles per second. Write that speed in scientific notation and use it in a calculation to estimate the distance light travels in an hour. Write the final answers in scientific notation format with the coefficient rounded to two decimal places. (*use calculator to multiply the coefficients*)

Section 3.3 Exponential Models

In chapter two application problems involving constant rates of change are modeled by arithmetic sequences and linear equations. In this chapter, application problems involving constant percentage changes are modeled by geometric sequences and exponential equations. A comparison of a simple interest savings account with a compounded annually savings account shows the difference between linear and exponential models. In a simple interest account, the interest earned each year is calculated by multiplying the principal, the original amount, times the annual interest rate. Each year a simple interest account earns the same amount of interest, with no interest earned from the interest gained in previous years. In a compounded annually account, the interest earned each year is calculated by multiplying the amount in the account at the beginning of each year times the annual interest rate. In compounded accounts, interest is earned not only from the principal but also from the interest gained in previous years.

Scenario 1 Sue deposits \$4000 in a savings account that earns 5% **simple interest**. Find the function $f(t)$ that gives balance after t years.

To find the slope, multiply the simple interest 5% per year times the principal \$4000

$$m = 5\%(\$4000) = \$200 \text{ per year}$$

The linear function $f(t) = 200t + 4000$ gives the balance in this simple interest savings account after t years. The amount in this account (balance) is an arithmetic sequence with initial value of \$4000 that increases by \$200 per year. The table below gives the balance in Sue's simple interest account during the first seven years.

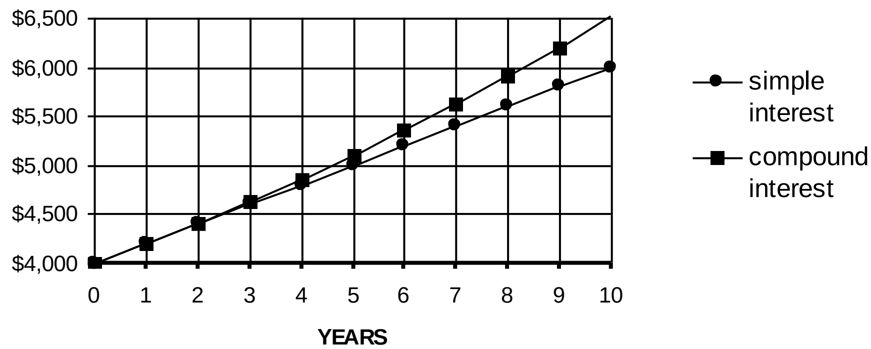
t years	0	1	2	3	4	5	6	7
$f(t)$ \$	4000	4200	4400	4600	4800	5000	5200	5400

Scenario 2 Sue deposits \$4000 in an account that earns 5% interest **compounded annually**. Find the function $f(t)$ that gives the balance after t years.

The balance in a compounded annually account is modeled by a geometric sequence with initial value of \$4000 that increases by 5% each year. The exponential function $f(t) = 4000(1.05)^t$ gives the balance in this compounded annually account after t years, since 100% + 5% = 105%. The table below gives the balance (*rounded to nearest dollar*) in Sue's compounded annually account during the first seven years.

t years	0	1	2	3	4	5	6	7
$f(t)$ \$	4000	4200	4410	4630	4862	5105	5360	5628

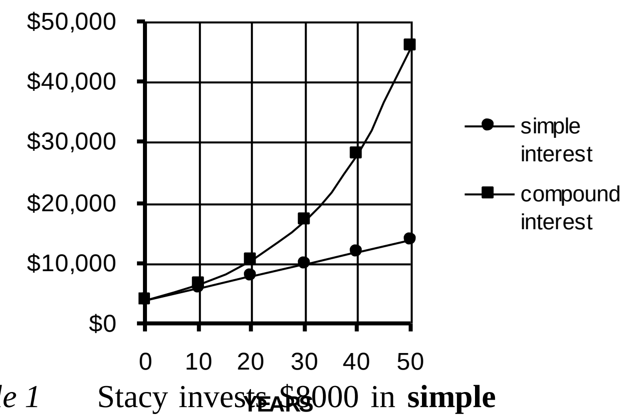
The graph below compares the amount in Sue's saving account depending on whether it was a simple interest or compound interest account over a ten-year period. A line with slope of $200 per year graphically represents the simple interest account while an exponential graph graphically represents the compounded account. Notice as the years pass the difference in the balances between the two types of accounts become greater as the effect of compounding which allows interested to be earned on previously gained interest becomes more and more prominent.



After significant time passes the balances in the compounded account will overwhelm the balances in the simple interest account. The table and graph below compares the balance in the accounts after ten, twenty, thirty, forty, and fifty years. After fifty years, the balance in the compounded account is more than three times as much as the simple interest account. Given enough time a **J** shaped exponential growth curve will overwhelm an increasing line.

Simple Interest	**Compounded annually**
$200(10) + 4000 = 6{,}000$	$4000(1.05)^{10} = 6{,}516$
$200(20) + 4000 = 8{,}000$	$4000(1.05)^{20} = 10{,}613$
$200(30) + 4000 = 10{,}000$	$4000(1.05)^{30} = 17{,}288$
$200(40) + 4000 = 12{,}000$	$4000(1.05)^{40} = 28{,}160$
$200(50) + 4000 = 14{,}000$	$4000(1.05)^{50} = 45{,}870$

Years invested	*Simple interest*	*Compound interest*
0	$4,000	$4,000
10	$6,000	$6,516
20	$8,000	$10,613
30	$10,000	$17,288
40	$12,000	$28,160
50	$14,000	$45,874



Example 1 Stacy invests $8000 in **simple**

interest account that earns 6% annual interest. How much money is in the account after four years?

To find the slope, multiply the simple interest 6% per year times the principal $8000

$$6\%(\$8000) = \$480 \text{ per year}$$

The linear function $f(t) = 480t + 8000$ gives the balance in this simple interest savings account after t years. The amount in this account (balance) is an arithmetic sequence with initial value of $8000 that increases by $480 per year. To find the balance $9920 after 4 years insert $t = 4$ into the function as shown below.

$$f(4) = 480(4) + 8000 = \$9920$$

Example 2 Luis invest $1200 in an annual compounded account that earns 4% annual interest. How much money is in the account after eight years?

This is a geometric sequence with initial value of $1200 that increases by 4% each year. The exponential function $f(t) = 1200(1.04)^t$ gives the amount in Luis' account after t years, since 100% + 4% = 104%. To find the balance of $1642 after 8 years, insert 8 into the function.

$$f(8) = 1200(1.04)^8 \approx \$1642$$

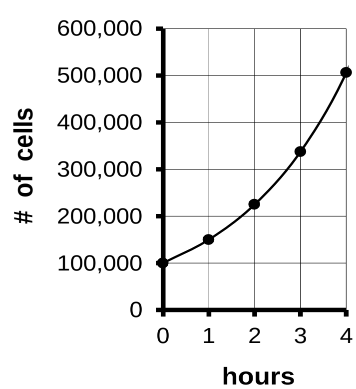
In many biological applications constant percentage growth is assumed during certain time periods. These applications are modeled using geometric sequences and exponential equations.

Example 3 100,000 cells are placed in an environment in which they increase by 50% each hour. Find the equation that models the number of cells.

This cell growth is modeled by a geometric sequence with an initial value of 100,000 cells that increases by 50% per hour. The number of cells after t hours is given by the exponential function $f(t) = 100{,}000(1.50)^t$ since 100% + 50% = 150%. The table below estimates the number of cells during the first four hours which when plotted forms the familiar **J** shaped exponential growth curve.

t hours	0	1	2	3	4
$f(t)$ cells	100,000	150,000	225,000	338,000	506,000

rounded to 3 significant digits



Example 4 Find the function that gives the number of bacteria in a culture which starts with 1000 bacteria and triples every hour.

$f(t) = 1000(3)^t$

This cell growth is modeled by a geometric sequence with an initial value of 1000 cells that triples in number every hour. The number of cells after t hours is given by $f(t) = 1000(3)^t$ since 100% + 200% = 300% this culture is increasing by 200% per hour.

Example 5 Find the function that gives the number of bacteria in a culture which starts with 5000 bacteria which increase by 24% each hour.

$f(t) = 5000(1.24)^t$

This cell growth is modeled by a geometric sequence with an initial value of 5000 cells that increases by 24% per hour. The number of cells after t hours is given by the exponential function $f(t) = 5000(1.24)^t$ since 100% + 24% = 124%.

Example 6 Find the function that gives the number of bacteria in a culture which starts with 5000 bacteria which decrease by 24% each hour.

$f(t) = 5000(0.76)^t$

This cell growth is modeled by a geometric sequence with an initial value of 5000 cells that decreases by 24% per hour. The number of cells after t hours is given by the exponential function $f(t) = 5000(0.76)^t$ since 100% – 76% = 24%.

Example 7 An economist predicts that inflation will grow by 3% each year for the next decade. Find the exponential equation that gives the relative value of current \$100 after t years of 3% inflation. What is the value of \$100 after six years of 3% percent inflation?

$f(t) = 100(0.97)^t$

Since inflation is growing by 3%, the relative value of money is decreasing by three percent each year. This relative value is modeled by a geometric sequence with an initial value of \$100 that decreases by 3% per year. The value of \$100 after t years is given by the exponential function $f(t) = 100(0.97)^t$ since 100% – 97% = 3%.

After six years of 3% inflation, \$100 has a relative value of \$83.30

$$f(6) = 100(.97)^6 \approx \$83.30$$

Example 8 A person drinks an espresso coffee that contains about 160 milligrams of caffeine and with each passing hour the amount of caffeine in their bloodstream decreases by approximately 11%. Find the function that gives the amount of caffeine in the bloodstream from the coffee t hours after drinking the coffee.

$f(t) = 160(0.89)^t$

This amount of caffeine is modeled by a geometric sequence with an initial value of 160 milligrams that decreases by 11% per hour. The caffeine in milligrams after t hours is given by the equation $f(t) = 160(0.89)^t$ since 100% – 11% = 89%. The table below estimates the number of cells during the first 6 hours which when plotted forms an inverted **J** shaped decreasing exponential curve.

t hours	0	1	2	3	4	5	6
$f(t)$ mg	160	142	127	113	100	89	80

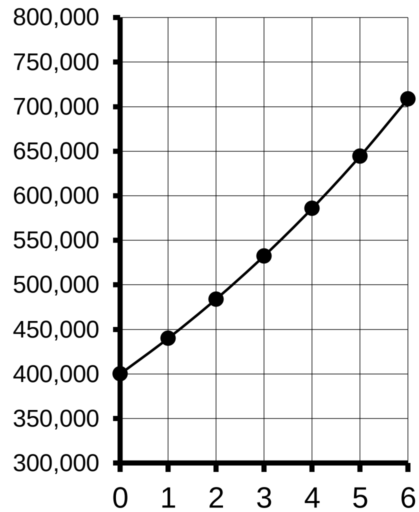
Example 9 The prices of homes in a Bay Area community are projected to increase in value by 10% each year. Ty owns a house in that community that is currently valued at $400,000. Find the function that gives the projected value of Ty's home t years from now. Make a table of values using one-year increments and plot these points on a grid. What is the projected value of Sam's home after eight years?

$f(t) = 400{,}000(1.10)^t$

The value of this home is modeled by a geometric sequence with an initial value of $400,000 which increases by 10% per year. The function $f(t) = 400{,}000(1.10)^t$ gives the value of the home after t years of 10% growth since 100% + 10% = 110%. To find the value of $857,436 after ten years insert 8 into the exponential function.

$$f(8) = 400{,}000(1.10)^8 \approx \$857{,}436$$

t years	0	1	2	3	4
$f(t)$ $	400,000	440,000	484,000	532,400	585,640



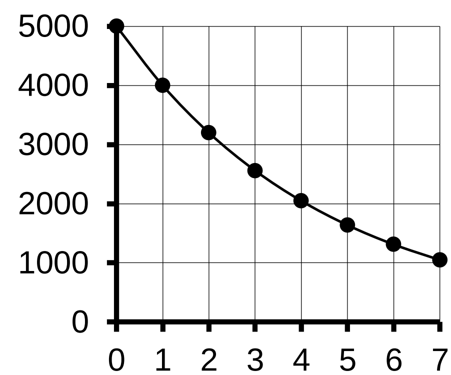
Example 10 A lake is polluted with 5000 tons of pollutants. An environmental research study calls for various water treatment projects that will remove 20% of the pollutants from the lake each year, while stopping any new pollution from reaching the lake. Find the function that gives the amount of tons of pollutants t years later if the remedies outlined by the study are followed. How many tons of pollutants would remain after ten years?

$f(t) = 5000(0.80)^t$

The pollution is modeled by a geometric sequence with an initial value of 5000 tons which decreases by 20% per year. The exponential function $f(t) = 5000(0.80)^t$ gives the amount of pollutants in lake t years later since 100% – 20% = 80%.

t years	0	1	2	3	4	5
$f(t)$ tons	5000	4000	3200	2560	2050	1640

3 significant digits



To find the tons of pollutants remaining after 10 years, insert $t = 10$ into the exponential formula as shown below. Ten years later, the lake would still have 536 tons of pollutants.

$$f(10) = 5000(.80)^{10} \approx 536 \text{ tons}$$

Example 11 Sal invest \$10,000 in an account for five years that returned 12% per year. Then Sal switched the entire balance into an account for three years which lost 4% per year. How much does Sal have in the account 8 years later?

The first account is a geometric sequence with an initial value of \$10,000 that increases by 12% per year with the balance in the account modeled by the exponential function $f(t) = 10{,}000(1.12)^t$ since 100% + 12% = 112%. After five years the amount in the account is \$17,623. The second account is a geometric sequence with an initial value of \$17,623 that decreases by 4% per year with the balance in the account modeled by the exponential function $g(t) = 17{,}623(0.96)^t$ since, 100% – 4% = 96%. As shown below the balance is \$15,592 after 8 years.

$$f(t) = 10{,}000(1.12)^t \qquad f(5) = 10{,}000(1.12)^5 \approx \$17{,}623$$
$$g(t) = 17{,}623(0.96)^t \qquad g(3) = 17{,}623(0.96)^3 \approx \$15{,}592$$

In the previous examples, problems with constant percentage increase or decrease per stage are modeled with an exponential function which is used to find the value after a given time period. Now these exponential functions are used to determine how much time is needed before a given value is reached. These problems are solved using a calculator as shown below. In the next chapter these same problems are solve algebraically using logarithms.

Example 12 Shanita deposits \$6000 in an annual compounded account that earns 5% annual interest. Estimate the years needed to double the initial amount by using a calculator to find balance for different years.

$f(t) = 6000(1.05)^t$

This is a geometric sequence with initial value of \$6000 that increases by 5% each year. The exponential function $f(t) = 6000(1.05)^t$ gives the amount in Shanita's account after t years, since 100% + 5% = 105%.

To estimate when the initial amount \$6000 doubles to \$12,000 use a calculator and find the balance after a reasonable number of years such as 10 years. Since this after 10 years it has not yet doubled, try 15 years. Using the information below, it takes approximately 15 years for the amount in the account to double.

$$f(10) = 6000(1.05)^{10} = \$9,773$$
$$f(15) = 6000(1.05)^{15} = \$12,474$$
$$f(14) = 6000(1.05)^{14} = \$11,880$$

Exercises 3.3

1. Ty invests $10,000 in a **simple interest account** with 5% annual interest.

A. Find the function $f(t)$ that gives the balance of this account after t years.

B. Make a table of values which gives the balance during the first five years.

2. Rachael invests $8000 in a **simple interest account** with 3.5% annual interest.

A. Find the function $f(t)$ that gives the balance of this account after t years.

B. Make a table of values which gives the balance during the first five years.

3. Tamara invests $10,000 in a compounded annually account that earns 5% annual interest.

A. Find the function $f(t)$ that gives the balance of this account after t years.

B. Make a table of values which gives the balance during the first five years.

C. Find the balance after eight years.

4. Jose invests $5000 in a compounded annually account that earn 3.5% annual interest.

A. Find the function $f(t)$ that gives the balance of this account after t years.

B. Make a table of values which gives the balance during the first five years.

C. Find the balance after six years.

5. Draymond invests $10,000 in a compounded annually account that earns 5% annual interest.

A. Find the function $f(t)$ that gives the balance of this account after t years.

B. Find the balance after 4.5 years.

C. Estimate the time in years needed to double the investment by using a calculator to find balance for different years. (*round to nearest year*)

6. Jolene is purchasing a house for $250,000 and estimates that the value will increase by 6% each year.

A. Find the function $f(t)$ that models the value of this house t years from now.

B. Estimate the value of this home ten years later.

C. Estimate the time in years when the value of the home will be $400,000 by using calculator to find balance for different years. (*round to nearest year*)

7. A small town has a population of 50,000 residents in 2016 and the population is projected to increase by 3% each year thereafter.

A. Find the function $f(t)$ that models the population of the town t years later.

B. Use the function to estimate the population of the town in 2025. (*round to 3 sig digits*)

C. Estimate the time in years before the population of this small town is 70,000 using a calculator to find population for different years. (*round to nearest year*)

8. The current world population in 2015 is 7.4 billion and is currently growing at the rate of 1.1% per year. Estimate populations in billions rounded to one decimal place.

A. Find the function $f(t)$ that models the world population t years later.

B. Estimate the population in 2050 using the current growth rate.

C. Estimate the population in 2100 using the current growth rate.

D. Estimate the population in 2100 assuming a growth rate of 1.6%

E. Estimate the population in 2100 assuming a growth rate of 0.6%

9. Sheila invested $5000 in a stock that averaged 5% earnings per year for the two years that she owned the stock. Then she switched the entire balance to an account that gained an average of 9% per year over the next three years.

A. Find the amount after initial two year period.

B. Find the amount in the investment after the five year period.

10. Jesse invested $10,000 in a stock that averaged 10% earnings per year for the four years he owned the stock. Then he switched the entire balance to an account which lost an average of 3% over the next two years.

A. Find the amount after initial four year period.
B. Find the amount in the investment after the six year period.

11. 40,000 cells are placed in an environment in which they increase in number by 15% each hour. Estimate number of cells to 3 significant digits.

A. Find the function $f(t)$ that models the number of cells after t hours.
B. Make a table of values that estimates the number of cells for each of the first four hours of the experiment.
C. Use the function to estimate the number of cells 1 day later.

12. 40,000 cells are placed in an environment in which they decrease in number by 25% each hour. Estimate number of cells to 3 significant digits.

A. Find the function $f(t)$ that models the number of cells after t hours.
B. Make a table of values that estimates the number of cells for each of the first four hours of the experiment.
C. Use the function to estimate the number of cells 12 hours later.

13. 100,000 cells are placed in an environment in which they decrease in number by 40% each hour. Estimate number of cells to 3 significant digits.

A. Find the function $f(t)$ that models the number of cells after t hours.
B. Use the function to estimate the number of cells 15 hours later.
C. Estimate the time in hours until the number of cells is half the original number using a calculator to find population for different hours. (*round to nearest year*)

14. 10,000 cells are placed in an environment in which they increase in number by 20% each hour. Estimate number of cells to 3 significant digits.

A. Find the function $f(t)$ that models the number of cells after t hours.
B. Use the function to estimate the number of cells 8 hours later.
C. Estimate the time in hours until the number of cells is double the original number using a calculator to find population for different hours. (*round to nearest year*)

Section 3.4 Exponential Models (doubling times & half-life)

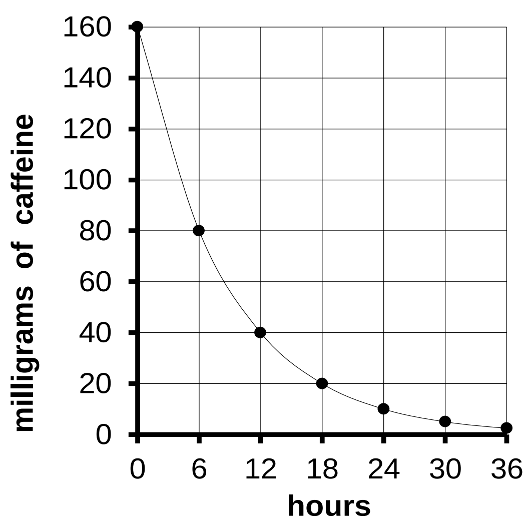
For exponential functions that model geometric sequences the growth/decay rate is given in terms of the percentage increase or decrease per stage. For these exponential functions there it is another measurement which for some applications serves as a better description than the percentage increase or decrease per stage. To illustrate this alternate approach an earlier problem is revisited.

Example 1 A person drinks an espresso coffee that contains about 160 milligrams of caffeine and the amount of caffeine in their bloodstream decreases by approximately 11% each hour. Find the function that measure the amount of caffeine in the blood stream after t hours and then make a table of values using 6, 12, 18, 24, 30, and 36 hours as the inputs.

$f(t) = 160(0.89)^t$

This amount of caffeine is modeled by a geometric sequence with an initial value of 160 milligrams that decreases by 11% per hour. The caffeine in milligrams after t hours is given by the function $f(t) = 160(0.89)^t$ since 100% – 11% = 89%. Inserting 6, 12, 18, 24, 30, and 36 hours as the inputs into this function resulting in the values in the following table which when plotted forms an inverted **J** shaped decreasing exponential curve. Looking at the table closely a pattern emerges, every 6 hours results in half of the milligrams of caffeine being removed from the bloodstream. So as alternative to describing the caffeine levels decreasing by 11% per hour it can be stated that the half-life of caffeine in the bloodstream is 6 hours.

t hours	$f(t)$ milligrams
0	160
6	$160(0.89)^6 \approx 80$
12	$160(0.89)^{12} \approx 40$
18	$160(0.89)^{18} \approx 20$
24	$160(0.89)^{24} \approx 10$
30	$160(0.89)^{30} \approx 5$
36	$160(0.89)^{36} \approx 2.4$



For applications involving exponential decay instead of defining the decay in terms of the percentage decrease it can be defined in terms of the half-life, which is the time required to lose half of its value. The function $f(t) = b(.50)^t$ is a geometric sequence with initial value b that decreases by 50% (half its value) each stage. To slow down this process so that the function is decreasing by 50% every h stages instead of every single stage, the exponent in this case the time t is divided by h.

Definition The function $f(t) = b(0.50)^{\left(t/h\right)}$ determines the population at time t with b the initial value and h the half-life time.

Example 2 A person drinks an espresso coffee that contains about 160 milligrams of caffeine. If caffeine in the bloodstream has a half-life of six hours, make a table that gives the milligrams of caffeine in this person bloodstream. Also find the exponential function that estimates the amount of caffeine in the bloodstream t hours later.

The following table is generated by starting with initial value of 160 milligrams and halving the milligrams of caffeine every 6 hours.

t hours	0	6	12	18	24	30	36
$f(t)$ mg	160	80	40	20	10	5	2.5

$f(t) = 160(0.50)^{\left(t/6\right)}$

The initial value is 160 milligrams and the half-life is 6 hours. The milligrams of caffeine after t hours are modeled by this exponential function.

Example 3 A chemical has a half-life of 30 years. Given that 100% of this chemical is initially present, find the function that gives the percentage of this chemical left after t years. Also find what percentage is left after 50 years?

The following table is generated by starting with initial value of 100 percent and halving the percentage of chemical left every 50 years.

t years	0	30	60	90	120
$f(t)$ %	100	50	25	12.5	6.25

$f(t) = 100(0.50)^{\left(t/30\right)}$

The initial value is 100% and the half-life is 30 years. To find that 31.5% of chemical is left after 50 years, insert 50 as the input and evaluate $f(50)$.

$$f(50) = 100(0.5)^{\left(50/30\right)} \approx 31.5\%$$

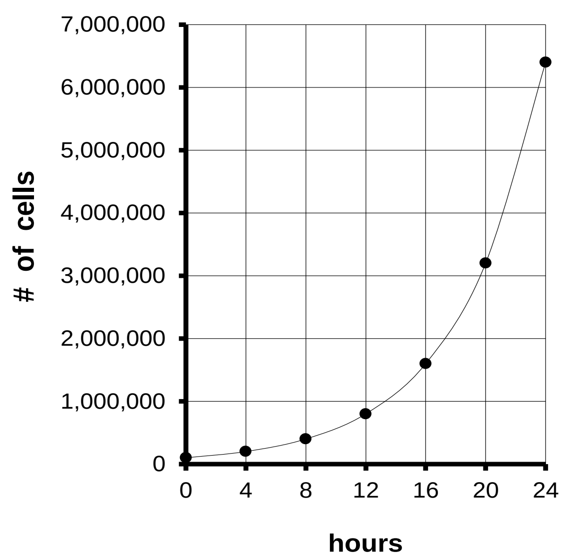
For applications involving exponential growth instead of defining the growth in terms of the percentage increase it can be defined in terms of the doubling time, which is the time required to double its value. The function $f(t) = b(2)^t$ is a geometric sequence with initial value b that decreases by 100% (doubles its value) each stage. To slow down this process so that the function is doubling every d stages instead of every single stage, the exponent in this case the time t is divided by d.

Definition The function $f(t) = b(2)^{(t/d)}$ determines $f(t)$ the population at time t with b the initial value and d the doubling time.

Example 4 100,000 cells are placed in an environment in which they double in number every four hours. Make an input/output table that gives the number of cells during a 24-hour time period and plot these values. Also, find the function that models the number of cells and use it to estimate the number of cells present two days later?

The initial value is 100,000 cells and the doubling time of 4 hours is used to generate the table below. The **J** shaped exponential growth curve models the cell population over a one-day period.

t hours	$f(t)$ # of cells
0	100,000
4	200,000
8	400,000
12	800,000
16	1,600,000
20	3,200,000
24	6,400,000



$$f(t) = 100{,}000(2)^{(t/4)}$$

The number of cells after t hours is modeled by the above exponential function since the initial value is 100,000 cells and the doubling time is 4 hours. The doubling time in this problem is measured in hours, so convert 2 days to 48 hours and insert the input 48 and evaluate $f(48)$ and round to 2 significant digits. Two days later there will be approximately 410 million cells.

$$f(48) = 100{,}000(2)^{(48/4)} \approx 410{,}000{,}000 \text{ cells}$$

Example 5 $4000 is invested in an account whose balance doubles every 9 years. Make a table of values, find the function that models the balance after t years, and use it to the balance in the account 15 years later.

The initial value of $4000 and a doubling time of 9 years generates the table below.

t years	0	9	18	27	36
$f(t)$ dollars	4000	8000	16000	32000	48000

$$f(t) = 4000(2)^{(t/9)}$$

The balance after t years is modeled by the above exponential function since the initial value is $4000 and the doubling time is 9 years. To find the balance after 15 years of $12,699 insert the input 15 and evaluate $f(15)$

$$f(15) = 4000(2)^{(15/9)} = \$12{,}699$$

In some exponential growth situations the doubling time gives more insight into a model and in other situations the percentage growth rate per stage is preferred. To determine the percentage increase given the doubling time, the doubling function is written in the percentage increase form $f(t) = b(a)^t$ with the use of a calculator.

Example 6 $4000 is invested in an account whose balance doubles every 9 years. Find the percentage increase rate per year.

$$f(t) = 4000(2)^{(t/9)}$$

The balance after t years is modeled by the above exponential function since the initial value is $4000 and the doubling time is 9 years. Write the exponent $t/9$ as the product of (1/9) time t as shown below and use a calculator to evaluate 2 raised to the 1/9 power (rounded to 4 significant digits). So doubling every 9 years is equivalent to approximately 8.0% increase per year, since the difference of 1.080 and 1 is 0.080 which equal 8.0%.

$$f(t) = 4000(2)^{(t/9)} = 4000(2)^{(1/9)(t)} \approx 4000(1.080)^t$$

Exercises 3.4

1. Ty invests \$10,000 in an account that doubles every 7 years.

A. Find the function $f(t)$ that gives the balance of this account after t years.

B. Use the doubling time to make a table that gives the balance in the account every 7 years without a calculator.

C. Find the balance after 12 years.

2. Debbie invests \$3000 in an account that doubles every 9 years.

A. Find the function $f(t)$ that gives the balance of this account after t years.

B. Use the doubling time to make a table that gives the balance in the account every 9 years without a calculator.

C. Find the balance after 15 years.

3. 20,000 cells are placed in an environment in which they double in number every five hours.

A. Use the doubling time to make a table that gives the number of cells every five hours without a calculator.

B. Find the function that models the number of cells after t hours.

C. Use this function to estimate the number of cells one day later. (*3 sign digits*)

4. 50,000 cells are placed in an environment in which they double in number every eight hours.

A. Use the doubling time to make a table that gives the number of cells every eight hours without a calculator.

B. Find the function that models the number of cells after t hours.

C. Use this function to estimate the number of cells 2 days later. (*3 sign digits*)

5. A given drug has a half-life of eight hours in a patient's bloodstream and a patient is injected with 120 milligrams of this medication.

A. Use the half-life to make a table the milligrams of medication in the patient every eight hours without a calculator.

B. Find the function that models the number of cells after t hours.

C. Use this function to estimate the number of cells 2 days later. (*2 sign digits*)

6. A given drug has a half-life of six hours in a patient's bloodstream and a patient takes 100 milligrams of this medication.

A. Use the half-life to make a table the milligrams of medication in the patient every six hours without a calculator.
B. Find the function that models the milligrams in bloodstream after t hours.
C. Use the function to estimate the drug amount in the bloodstream 1 day later. (*2 sign digits*)

7. A aspirin tablet has a half-life of approximately 15 minutes and a patient takes a tablet with 80 milligrams.

A. Use the half-life to make a table the milligrams of medication in the patient every 15 minutes without a calculator.
B. Find the function that models the number of cells after t minutes.
C. Use this function to estimate the number of cells 2 hours later. (*2 sign digits*)

8. Hydrogen peroxide has a half-life of approximately 15 hours in air.

A. Given that 100% of hydrogen peroxide is initially present, use the half-life to make a table of the percentage of hydrogen peroxide every 15 hours in air.
B. Find the function that models the hydrogen peroxide left after t hours is air.
C. Use this function to estimate percentage hydrogen peroxide after 3 days in air. (*2 sign digits*)

9. Carbon-14 has a very long half-life of 5730 year and is often used to estimate the age of ancient finds.

A. Given that 100% of this carbon-14 is initially present, use the half-life to make a table of the percentage of carbon-14 left every 5730 years without a calculator.
B. Find the function that models the percentage of carbon-14 after t years.
C. Use this function to estimate percentage of carbon-14 after 100 years.
D. Use this function to estimate percentage of carbon-14 after 20,000 years. (*2 sign digits*)

Section 3.5 Exponential Models (good fit exponentials)

In this section a technique is developed to find an exponential model that goes through the y*-intercept and another point. Before proceeding to that technique below is a reminder of how a linear model is found that goes through the* y*-intercept and another point.*

Definition The **slope** or the **average rate of change** of the line connecting the two points $(x_1\ ,\ y_1)$ and $(x_2\ ,\ y_2)$ is denoted by the letter m and defined by

$$m \;=\; \frac{rise}{run} \;=\; \frac{(y_2 - y_1)}{(x_2 - x_1)}$$

	x	y	
	x_1	y_1	
run	x_2	y_2	**rise**

Example 1 *Find linear function* f*(*x*)* = mx + b *that connects the following two points the* y*-intercept (0 , 20) and (5 , 50)*

Since the y-intercept is given, $f(x) = mx + 20$
To find the slope use the slope formula as shown below

$$m \;=\; \frac{(y_2 - y_1)}{(x_2 - x_1)} \;=\; \frac{50 - 20}{5 - 0} \;=\; \frac{30}{5} \;=\; 6$$

$f(x) = 6x + 20$ with slope 6 and y-intercept (0 , 20)

This function generates an arithmetic sequence with initial value of 20 that increases by 6 units per stage. To graph the line start at the point (0 , 20) and go six units up for every one unit across. To check that the other point (5 , 50) is located on this line, show that $f(5) = 50$.

$$f(5) \;=\; 6(5) + 20 \;=\; 30 + 20 \;=\; 50$$

To find the linear function connecting two points the slope formula needed which measures the constant increase or decrease. To find the exponential function connecting two points a similar type formula is needed which measures not the constant increase or decrease but the common ratio which gives the percentage increase or decrease.

Definition The **common ratio** of the exponential function connecting the y-intercept point $(\ 0\ ,\ y_1\ )$ and another point $(\ x_2\ ,\ y_2\ )$ is denoted by the letter a and defined by

$$(a)^{x_2 - x_1} \quad = \quad (y_2 / y_1)$$

Example 2 Find exponential function $g(x) = b(a)^x$ that connects the following two points the y-intercept (0 , 20) and (5 , 50)

Since the y-intercept is given, $g(x) = 20(a)^x$
To find the base a, use the common ratio formula listed below

$$(a)^{x_2 - x_1} \quad = \quad (y_2 / y_1)$$

$$a^5 \quad = \quad (5/2)$$

Raise each side to the 1/5 power (taking the 5th root of each side)

$$a \quad = \quad (5/2)^{(1/5)} \;\approx\; 1.2011 \qquad (\textit{round to 4 decimal places})$$

$g(x) \quad = \quad 20(1.201)^x$

This function generates a geometric sequence with initial value of approximately 20 that increases by 20.11% per stage, since 1.2011 – 1.00 = 0.2011 which equals 20.11%. To check that the other point (5 , 50) is located on this exponential graph, show that $f(5) \approx 50$.

$$f(5) = 20(1.2011)^5 \approx 49.97$$

Example 3 Find the **linear function** that models the value of a home was appraised at \$250,000 in 2004 and in \$330,000 in 2012. Use this linear function to estimate the value in 2016.

Let 2004 be the base year with t representing the years since 2004. The information can be written as the points (0 , 250000) and (8 , 330000). Use the slope formula to find average increase per year.

$$m = \frac{(y_2 - y_1)}{(x_2 - x_1)} = \frac{330,000 - 250,000}{8 - 0} = \frac{80,000}{8} = \$10,000 \text{ per year}$$

$f(t) = 10,000t + 250,000$

The initial value of the home is \$250,000 and it increases in value by \$10,000 per year. For value in 2016, evaluate $f(12)$. The estimate value in 2016 is \$370,000.

$$f(12) = 10,000(12) + 250,000 = \$370,000$$

Example 4 Find the **exponential function** that models the value of a home was appraised at \$250,000 in 2004 and in \$330,000 in 2012. Use this exponential function to estimate the value in 2016.

Let 2004 be the base year with t representing the years since 2004. The information can be written as the points (0 , 250000) and (8 , 330000). First use the common ratio formula to find the base a.

$$(a)^{x_2 - x_1} = (y_2 / y_1)$$

$$a^8 = (330,000/250,000)$$

$$a^8 = (33/25)$$

$$a = (33/25)^{(1/8)} \approx 1.0353 \quad (\textit{4 decimal places})$$

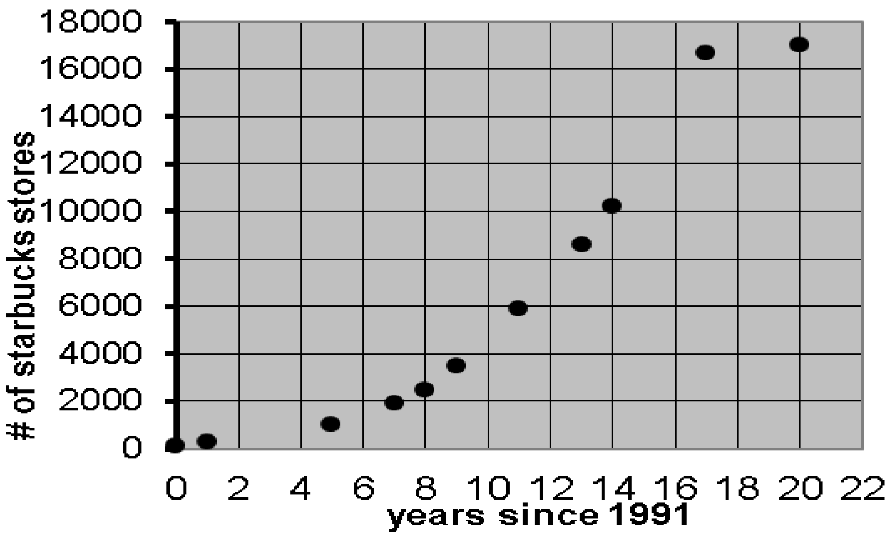
$f(t) = 250,000(1.0353)^t$

The initial value of the home is \$250,000 and it increases by 3.53% each year, since 1.0353 – 1 = 0.0353 which equals 3.53%. To find the value in 2016, evaluate $f(12)$. The estimate value in 2016 is \$379,000

$$f(12) \;=\; 250{,}000(1.0353)^{12} \;\approx\; \$379{,}000$$

Example 5 Below is a table which gives $f(t)$ number of Starbuck store with t representing the years since 1991. Find the good fit exponential function that models this data using the two bolded data points. Then evaluate $f(30)$ and describe in sentence form what this means including appropriate units. Also, predict the number of stores in 2018 and compare it with the actual number of stores.

t years	**0**	1	5	7	8	9	11	13	14	17	**20**
$f(t)$ stores	**120**	270	1020	1890	2490	3500	5890	8570	10240	16680	**17010**



To find the base a, use the common ratio formula. Round a to four decimal places

$$(a)^{x_2 - x_1} \;=\; \left(y_2 / y_1\right)$$
$$a^{20} \;=\; (17010/120)$$
$$a \;=\; (17010/120)^{(1/20)} \;\approx\; 1.2811 \quad (\textit{4 dec places})$$

$f(t) \;=\; 120(1.2811)^t$

The initial value is 120 stores in 1991 and the number of stores is increasing by 28.11% each year, since 1.2811 – 1 = 0.2811 which equals 28.11%

Below $f(30)$ is evaluated and rounded to 3 significant figures. Since 1991 + 30 = 2021, this model predicts that in 2021 there will be around 203,000 Starbuck stores.

$$f(30) \;=\; 120(1.2811)^{30} \;\approx\; 203{,}000 \text{ stores}$$

To predict the number of stores in 2018 evaluate $f(27)$ since 2018 – 1991 = 27. As shown below, the model predicts that there will be 96,400 stores in 2018.

$$f(27) \;=\; 120(1.2811)^{27} \;\approx\; 96{,}400 \text{ stores}$$

Exercises 3.5

1-4. Use the common ratio formula to find the exponential function $f(t) = b(a)^t$ that goes through the following two given points. Round a to four decimal places. Also determine the percentage increase (or decrease) per stage.

1. (0 , 20) and (5 , 30)

2. (0 , 100) and (12 , 735)

3. (0 , 20) and (9 , 12)

4. (0 , 100) and (10 , 43.5)

5-6. Use the common ratio formula to find the "good fit" exponential function $f(t) = b(a)^t$ that that goes through the two bolded points in the table and use it to answer the following questions. Round a to four decimal places.

5.

t	**0**	2	4	6	**8**	10
$f(t)$	**5.2**	7.1	9.0	11.3	**15.7**	20.3

5A. Find the "good fit" exponential function $f(t)$

5B. Find the percentage increase (or decrease) per stage.

5C. Evaluate $f(5)$ Is this interpolation or extrapolation?

5D. Evaluate $f(15)$ Is this interpolation or extrapolation?

6.

x	**0**	3	6	9	**12**
$f(t)$	**273**	205	173	140	**121**

6A. Find the "good fit" exponential function $f(t)$

6B. Find the percentage increase (or decrease) per stage.

6C. Evaluate $f(20)$ Is this interpolation or extrapolation?

6D. Evaluate $f(10.5)$ Is this interpolation or extrapolation?

7&8. The federal government performs a census every ten years and found the population of the city of Fairfield in 2000 is 96,200 and in 2010 is 105,300.

7A. Assume that the population of Fairfield is modeled by **linear growth.** Find the find the **slope** of the good fit line (*round to nearest ten*) using the two given data points. Describe in **sentence form** how this slope relates to this problem. Include appropriate units.

7B. Find the linear function $f(t)$ which estimates the population of Fairfield t years after 2000.

7C. Use $f(t)$ to estimate the population of Fairfield in 2025.

7D. Use $f(t)$ to estimate the population of Fairfield in 2050.

8A. Assume that the population of Fairfield is modeled by **exponential growth.** Use the common ratio formula to find the "good fit" exponential function $f(t) = b(a)^t$ which estimates the population of Fairfield t years after 2000 using the two given data points. Round a **to four decimal places**

8B. Find the percentage increase in population in Fairfield.

8C. Use function $f(t)$ to estimate the population of the city Fairfield in 2025.

8D. Use function $f(t)$ to estimate the population of Fairfield in 2050.

9. The table below gives $f(t)$ the U.S. national debt in trillions with t representing the years since 1960.

t years	**0**	10	20	30	40	**50**
$f(t)$ trillions	**0.29**	0.39	0.93	3.2	6.0	**14.1**

9A. Use the common ratio formula to find the "good fit" exponential function $f(t) = b(a)^t$ which estimates the U.S. national debt in trillions t years since 1960. Round a **to four decimal places**

9B. Find the percentage increase in national debt.

9C. Find $f(60)$ and describe in sentence form what this means including appropriate units.

9D. Use function $f(t)$ to predict the US national debt in 2016 and compare it with the actual national debt in 2016.

9E. Use function $f(t)$ to predict the US national debt in 2024.

10. The table below gives $P(t)$ the world population in billions with t representing the years since 1950.

t years	**0**	10	20	30	40	50	**60**
$P(t)$ billions	**2.5**	3.0	3.7	4.5	5.3	6.1	**6.9**

10A. Use the common ratio formula to find the "good fit" exponential function $P(t) = b(a)^t$ which estimates the world population in billions t years since 1960. Round a **to four decimal places**

10B. Find the percentage increase in world population.

10C. Find $P(100)$ and describe in sentence form what this means including appropriate units.

10D. Use function $P(t)$ to predict the population in 2016 and compare it with the actual world population in 2016.

Section 3.6 Compound Interest

Two types of interest problems **simple interest** and **compounded annually interest** have been covered earlier. After a review of these two types of interest problems, a formula is derived for compounded interest applications in which the period of compounding is more often than once per year.

Example 1 Jose invests \$5000 in a simple interest account that earns 8% annual interest. Find the function that models the amount in the account. Find the amount after ten years.

The principal $P = 5000$

The slope m = (rate)(principal) = 8%(\$5000) = \$400 per year

$f(t) = 400t + 5000$

This function generates an arithmetic sequence with an with initial value \$5000 that increases by \$400 per year.

t	0	1	2	3	4	5
$f(t)$	5000	5400	5800	6200	6600	7000

Evaluate $f(10)$ as shown below to show that \$9000 is in the account after ten years.

$$f(10) = 400(10) + 5000 = \$9000$$

Example 2 Whitney invests \$5000 in a compounded annually account that earns 8% annual interest. Find the function that models the amount in the account. Find the amount after ten years.

The principal (initial value) is \$5000 and the amount increases by 8% each year.

$f(t) = 5000(1.08)^t$

This is a geometric sequence with initial value \$5000 multiplied by 1.08 each year.

t	0	1	2	3	4	5
$f(t)$	5000.00	5400.00	5832.00	6298.56	6802.44	7346.64

Evaluate $f(10)$ as shown below to show that \$10,745 is in the account after ten years.

$$f(10) = 5000(1.08)^{10} = \$10,794.62$$

Consider what happens when Whitney invests \$5000 in an account that earns 8% annual interest which is **compounded quarterly**. Since the compounding is done four times each year, each quarter the interest earned is only 2%, which is the 8% annual interest rate divided by four, and each year the compounding process of earning interest on previously earned interest is done four times.

After one year, four quarterly compounding periods of 2% have to be calculated.

$$f(1) = 5000(1 + .08/4)^4 = \$5412.16$$

After two years, eight quarterly compounding periods of 2% have to be calculated.

$$f(2) = 5000(1 + .08/4)^8 = \$5858.30$$

After three years, twelve quarterly compounding periods of 2% have to be calculated.

$$f(3) = 5000(1 + .08/4)^{12} = \$6341.21$$

After four years, sixteen quarterly compounding periods of 2% have to be calculated.

$$f(4) = 5000(1 + .08/4)^{16} = \$6863.93$$

t	0	1	2	3	4
$f(t)$	5000.00	5412.16	5858.30	6341.21	6863.93

Definition **Interest compounded *n* times per year** occurs when interest is applied n times per year to all the money in the account including the interest previously earned. This is modeled by the exponential function, $f(t) = P(1 + r/n)^{(nt)}$ which gives the amount $f(t)$ in the account after t years with principal P and annual interest rate r which is compounded n times each year.

In the compounding formula $f(t) = P(1 + r/n)^{(nt)}$ the number of compounding periods n per year serves two roles, it divides into the annual interest rate r to represent the rate r/n earned each compounding period and is multiplied times the number of years t to indicate that nt compounding periods that occur in t years.

Example 3 Tyrone invests \$5000 in an account that earns 8% annual interest that is compounded quarterly. Find the function that models the amount in the account. Find the amount after ten years.

Use compound interest formula with $P = 5000$, $r = .08$, and $n = 4$

$f(t) = 5000\left(1 + .08/4\right)^{(4t)}$ *leave function in this form do not divide .08 by 4*

The amount after ten years is given by $f(10) = 5000(1 + .08/4)^{40} \approx \$11,040$

enter the entire line into your calculator

Example 4 Maria invests \$10,000 in an account that earns 5% annual interest that is compounded weekly. Find the function that models the amount in the account. Find the amount after ten years.

Use the compound interest formula with $P = 10,000$ $r = .05$ and $n = 52$

$f(t) = 10,000\left(1 + .05/52\right)^{(52t)}$ *leave function in this form do not divide .05 by 52*

The amount after ten years is given by $f(10) = 10,000(1 + .05/52)^{520} \approx \$16,483$

enter the entire line into your calculator

Example 5 Lisa invests \$20,000 in an account that earns 4% annual interest that is compounded monthly. Find the function that models the amount in the account. Find the amount after six months.

Use the compound interest formula with $P = 20,000$ $r = .04$ and $n = 12$

$f(t) = 20,000\left(1 + .04/12\right)^{(12t)}$ *leave function in this form do not divide .04 by 12*

Be careful, six months is equal to ½ year

The amount after six months is given by $f(0.5) = 20,000(1 + .04/12)^{6} \approx \$20,403$

enter the entire line into your calculator

When the compounded interest formula is calculated with $n = 1$ the resulting expression simplifies to the exponential formula $f(t) = a(b)^t$ used in exponential growth problems earlier this chapter with b representing the principal P (initial value) and a equal to sum of 1 and annual interest rate r.

$$f(t) = P(1 + r/1)^{(1t)} = P(1 + r)^t = b(a)^t$$

In the previous examples, the present value P (initial value) and interest rate is given and the future value at a given time represented by the appropriate exponential function is calculated. In the following examples this process is reversed, with the future value at a specified time and interest rate given and the present value P required to generate that future value is calculated.

Example 6 Find the present value required so that if an account increases by 6% per year after 12 years the future value is \$100,000

$f(t) = P(1.06)^t$

Present value (initial value) P is unknown and the account increase by 6% each year. After 12 years the value is \$100,000 is written in function form as $f(12) = 100{,}000$ which is solved below to find P. As shown below \$49,697 is present value.

$$f(12) = P(1.06)^{12} = \$100{,}000$$

To solve for P divide both sides by $(1.06)^{12}$

$$P = 100{,}000/(1.06)^{12} \approx \$49{,}697$$

Example 7 Shondelle plans to purchase a car in 5 years with a down payment of \$5000. How much does she have to deposit in an account now that returns 3% compounded weekly so that she has enough in the account for the down payment.

Use the compound interest formula with $r = .03$ and $n = 52$

$f(t) = P(1 + .03/52)^{(52t)}$

After 5 years the value is \$5000 is written in function form as $f(5) = 5000$ which is solved below to find P. As shown below, Shondelle needs to deposit \$4304 not (present value) so that her account grows to \$5000 in 5 years.

$$f(5) = P(1 + .03/52)^{260} = \$5000$$

To solve for P divide both sides by $(1 + .03/52)^{260}$

$$P = 5000/(1 + .03/52)^{260} \approx \$4304$$

Exercises 3.6

1A. Aaron invests \$10,000 in a **simple interest** account that earns 6% annual interest. Find the function that models the amount in the account.

$f(t)$ = ____________________

1B. Find the amount after seven years.

$f(7)$ = ____________________ = \$ ______________

2A. Barbara invests \$80,000 in a **simple interest** account that earns 4% annual interest. Find the function that models the amount in the account.

$f(t)$ = ____________________

2B. Find the amount after twelve years.

$f(12)$ = ____________________ = \$______________

3A. Curtis invests \$20,000 in a **compounded annually** account that earns 5% annual interest. Find the function that models the amount in the account.

$f(t)$ = ____________________

3B. Find the amount after eight years.

$f(8)$ = ____________________ = \$______________

4A. Devon invests \$800 in an account that earns 4% annual interest that is **compounded monthly**. Find the function that models the amount in the account. (*do not simplify inside parenthesis*)

$f(t)$ = ____________________

4B. Find the amount after twenty years.

$f($ ___ $)$ = __________________________ = \$____________

5A. Erica invests \$120,000 in an account that earns 7% annual interest that is **compounded weekly**. Find the function that models the amount in the account. (*do not simplify inside parenthesis*)

$f(t)$ = ____________________

5B. Find the amount after five years.

$f($ __ $)$ = ______________________ = $__________

6A. Fran invests $4000 in an account that earns 9% annual interest that is **compounded daily.** Find the function that models the amount in the account. (*do not simplify inside parenthesis*)

$f(t)$ = ____________________

6B. Find the amount after ten years.

$f($ __ $)$ = ______________________ = $__________

7A. George invests $27,000 in an account that earns 6% annual interest that is **compounded quarterly**. Find the function that models the amount in the account. (*do not simplify inside parenthesis*)

$f(t)$ = ____________________

7B. Find the amount after **six months**.

$f($ __ $)$ = ______________________ = $__________

8. How much does Ashley need to invest now in an account that pays 4% annual interest compounded annually so that in six years the balance of the account in $4000?

9. Juan invests in a stock account which he hopes returns an average of 7% per year compounded annually during the next five years. If his goal is for the stock account to have a balance of $10,000 in five year, how much would Juan need to invest in the account now?

10. How much does Ashley need to invest now in an account that pays 4% annual interest compounded monthly so that in six years the balance of the account in $4000?

11. Sheila plans to purchase a home in 4 years with a down payment of \$15,000. How much does she have to deposit in an account now that returns 5% compounded daily so that she has enough in the account for the down payment.

12A. For an account with a principal of \$5000 that earns 8% annual interest for each of the following compounding periods find the balance after 10 years and round $f(10)$ to the nearest dollar.

Monthly $n =$ _____

$f(10) =$ ______________________________ = \$__________

____

Weekly $n =$ _____

$f(10) =$ ______________________________ = \$__________

____

Daily $n =$ _______

$f(10) =$ ______________________________ = \$__________

____

Hourly $n =$ ____________

$f(10) =$ ______________________________ = \$__________

____

Minutely $n =$ ________________

$f(10) =$ ______________________________ = \$__________

____

12B. The Exponential (Euler number) denoted by the letter e is defined by what happens to the expression $(1 + 1/n)^n$ as the value of n gets larger and larger and goes toward infinity. Fill out the following table to estimate the value of the exponential number e. Round your answers to 4 decimal places

n	1	10	100	10,000	1,000,000
$(1 + 1/n)^n$					

12C. Find the value of Euler number e by using your calculator. The e button is located on most calculators above the ln button (using the shift button). Find the value of e by entering e raised to the first power, in most calculators e^1 (round the answer to 4 decimal places). Compare the value given by the calculator with the last line in the table in #12B

$e \approx$ ___________

Continuous (growth) interest model calculates what happens as the interest is compounded more and more times per year, such as every second, milli-second, … , instantaneously. This concept in used to find the amount as the number of times the growth rate is compounded per years gets larger and larger and goes toward infinity and in effect we are compounding instantaneously.

Definition The amount $f(t)$ in an account **compounded continuously** after t years with principal P and annual interest rate r is given by the exponential function, $f(t) = P(e)^{(rt)}$

13A. $5,000 is invested in an account that earns 8% annual interest that is **compounded continuously**. Find the function that models this account

$f(t) =$ _______

____

13B. Find the amount after ten years.

$f(10) =$ ___________ =

$___________

14. Compare the answer to 13B and the 12A when compounded minutely. How are these related?

Section 3.7 Annuities & Mortgages

In previous sections an exponential function $f(t) = b(a)^t$ which models a geometric sequence is used to find the future value of a compound interest account given the initial deposit (principal), the interest rate and compounding period. Now instead of having only one initial deposit and finding the future value, equal monthly deposits are made and the resulting future value is found. To find the future value of these equal monthly deposits, a formula is needed for the sum of first k terms of a geometric sequence. Note, since the first term listed is the initial value which is the value of the geometric sequence at the zero stage (initial stage), the list of first k terms starts with the initial stage and goes until the k-1 stage (not the k stage).

Notation The notation for the sum of the first k terms of the function $f(x)$ is written below using the Greek uppercase letter sigma ∑

$$\sum_{0}^{k-1} f(x) = f(0) + f(1) + f(2) + f(3) + \ldots + f(k-1)$$

Example 1 Use summation notation to list the first k terms of the geometric sequence $f(t) = b(a)^t$ with initial value b multiplied by a at each stage.

$$\sum_{0}^{k-1} f(x) = f(0) + f(1) + f(2) + f(3) + \ldots + f(k-1)$$

$$\sum_{0}^{k-1} b(a)^x = ba^0 + ba^1 + ba^2 + ba^3 + \ldots + ba^{k-1} \qquad \text{Use } f(t) = b(a)^t$$

$$\sum_{0}^{k-1} b(a)^x = b + ba + ba^2 + ba^3 + \ldots + ba^{k-1} \qquad \text{Simplify}$$

$$\sum_{0}^{k-1} b(a)^x = b[1 + a + a^2 + a^3 + \ldots + a^{k-1}] \qquad \text{Common factor of } b$$

Example 2 Use summation notation to list the first five terms of the geometric sequence $f(t) = 100(1.20)^t$

Since the initial value of a geometric sequence is located at the zero stage to find the sum of the first five terms of a sequence start with the initial stage $f(0)$ and go until the fourth stage $f(4)$ as shown below.

$$\sum_{0}^{4} 100(1.20)^x = 100(1.20)^0 + 100(1.20)^1 + 100(1.20)^2 + 100(1.20)^3 + 100(1.20)^4$$

$$\sum_{0}^{4} 100(1.20)^x = 100(1) + 100(1.20) + 100(1.20)^2 + 100(1.20)^3 + 100(1.20)^4$$

$$\sum_{0}^{4} 100(1.20)^x = 100[1 + 1.20 + 1.20^2 + 1.20^3 + 1.20^4] = 100(7.4416) = 744.16$$

Since some applications involve finding the sum of the first 360 terms of a geometric sequence listing out all the terms is not practical. A formula is needed that finds the summation of the first k terms of a geometric sequence. A clever technique that derives the formula for sum of the first k terms of a geometric sequence $f(t) = b(a)^t$ is to list the sum of the first k terms of a geometric sequence and also list a times the sum of the first k terms of this same geometric sequence and then subtract these two lists. As shown below taking the difference of these summations, results in all the terms canceling out (shown in bold) in the lists except the first and last term.

$$a\sum_{0}^{k-1} b(a)^x = ab[1 + a + a^2 + a^3 + \dots + a^{k-1}] = b[\boldsymbol{a} + \boldsymbol{a^2} + \boldsymbol{a^3} + \dots + \boldsymbol{a^{k-1}} + a^k]$$

$$-\sum_{0}^{k-1} b(a)^x = b[1 + a + a^2 + a^3 + \dots + a^{k-1}] = b[1 + \boldsymbol{a} + \boldsymbol{a^2} + \boldsymbol{a^3} + \dots + \boldsymbol{a^{k-1}}]$$

$$a\sum_{0}^{k-1} b(a)^x - \sum_{0}^{k-1} b(a)^x = b[a^k - 1] \qquad \text{Difference of two sums}$$

$$(a-1)\sum_{0}^{k-1} b(a)^x = b[a^k - 1] \qquad \text{Common factor } \sum_{0}^{k-1} b(a)^x$$

$$\sum_{0}^{k-1} b(a)^x = b(a^k - 1)/(a - 1) \qquad \text{Divide both side by } (a - 1)$$

Formula The sum of the first k terms of the geometric sequence $f(t) = b(a)^t$ stage is given by the **geometric sequence summation formula** with b the initial value and a the constant number multiplied by at each stage.

$$\sum_{0}^{k-1} b(a)^x = b((a)^k - 1)/(a - 1)$$

Example 3 Use summation formula to list the first five terms of the geometric sequence $f(t) = 100(1.20)^t$

$$\sum_{0}^{4} 100(1.20)^x = 100((1.20)^5 - 1)/(1.20 - 1) = 100((1.20)^5 - 1))/(.20) = 744.16$$

Example 4 Use summation formula to list the first fifteen terms of the geometric sequence $f(t) = 2000(1.05)^t$

$$\sum_{0}^{14} 2000(1.05)^x = 2000((1.05)^{15} - 1)/(1.05 - 1) = 2000((1.05)^{15} - 1)/(.05) \approx 43{,}157$$

The summation formula for a geometric sequence serves as a model for the balance of an account when regular equal size deposits are made for an extended period of time. Since the most common time period is monthly deposits, for the remainder of this section all formulas are based on monthly deposits (or payments) with the interest compounded monthly.

Scenario 1 To save for a down payment on a car, Mario deposits \$150 in an account which earns 2% annual interest compounded monthly for three years. Find the balance in this account after three years.

To make this scenario easier to model assume that Mario makes this commitment on Jan. 1, 2017 and makes the first deposit on Feb. 1, 2017. Below is the future value of each deposit three years later on Jan. 1, 2020.

First deposit on Feb. 1, 2017 will be in the account for 35 months
(not 36 months since Mario starts making deposits after first month)
$$150(1 + .02/12)^{35}$$

Second deposit on March 1, 2017 will be in the account for 34 months
$$150(1 + .02/12)^{34}$$

Third deposit on April 1, 2017 will be in the account for 33 months
$$150(1 + .02/12)^{33}$$

Now jumping to the last three of the 36 deposit made by Mario
Third to last payment on Nov. 1, 2019 will be in the account for 2 months
$$150(1 + .02/12)^{2}$$

Second to last payment on Dec. 1, 2019 will be in the account for 1 month
$$150(1 + .02/12)^{1}$$

Last payment on Jan. 1, 2020 will be in the account for 0 month
$$150(1 + .02/12)^{0} = 150$$

The future value on Jan. 1, 2020 of the 36 deposits is the following sum which lists the future value of the last payment first and goes backward until the first payment which is compounded 35 times. This sum consists of first 36 terms of the geometric sequence $f(t) = 150(1 + .02/12)^t$ is shown below then written using summation notation and evaluated using the summation formula.

$$150 + 150(1+.02/12)^1 + 150(1+.02/12)^2 + \dots + 150(1+.02/12)^{33} + 150(1+.02/12)^{34} + 150(1+.02/12)^{35}$$

$$\sum_{0}^{35} 150(1 + .02/12)^x = 150((1+.02/12)^{36} - 1)/(1+.02/12 - 1) = 150((1+.02/12)^{36} - 1)/(.02/12) \approx \$5561$$

Looking at the summation formula written below used to solve the previous problem, 150 represents the monthly deposit amount, .02/12 is the annual interest rate of 2% divided by 12 months in a year, and the 35 indicates the sum of the first 36 terms of the sequence with 36 equal to the number of months in 3 years.

$$\sum_{0}^{35} 150(1 + .02/12)^x$$

Since all the applications in this section will involve monthly payments with interest compounded monthly, the summation formula for a geometric sequence is written below with M the equal monthly deposits, r the annual interest rate, and t the number of years. Since there are 12 months each year, the future value of these deposits will result in a list with $12t$ terms. Below the geometric sequence summation formula is written for the function $f(t) = M(1 + r/12)^{12t}$ with $b = \boldsymbol{M}$ and $a = \mathbf{1 + \boldsymbol{r}/12}$

$$\sum_{0}^{k-1} b(a)^x \quad = \quad b[a^k - 1]/(a-1) \qquad \text{Replace with } b = \boldsymbol{M} \text{ and } a = \mathbf{1 + \boldsymbol{r}/12}$$

$$\sum_{0}^{12t-1} M(1+r/12)^x \quad = \quad \boldsymbol{M}((1+\boldsymbol{r}/\mathbf{12})^{12t} - 1)/(1 + \boldsymbol{r}/\mathbf{12} - 1)$$

$$\sum_{0}^{12t-1} M(1+r/12)^x \quad = \quad \boldsymbol{M}((1+\boldsymbol{r}/\mathbf{12})^{12t} - 1)/(\boldsymbol{r}/\mathbf{12}) \qquad \text{Reciprocal of denominator}$$

$$\sum_{0}^{12t-1} M(1+r/12)^x \quad = \quad (\boldsymbol{M})(\mathbf{12})((1+\boldsymbol{r}/\mathbf{12})^{12t} - 1)/(\boldsymbol{r})$$

Formula The future value of an account after t years in which the amount M is deposited each month which is compounded monthly with an annual interest rate r is determined by the summation formula below.

$$\sum_{0}^{12t-1} M(1+r/12)^x \quad = \quad (M)(12)((1+r/12)^{12t} - 1)/(r)$$

Example 5 To save for a down payment for a home, Tre deposits \$300 in an account which earns 4% annual interest compounded monthly for five years. Find the balance in this account after five years.

Use the summation formula with $M = 300$, $r = .04$ and $t = 5$

$$\sum_{0}^{59} 300(1+.04/12)^x \quad = \quad (300)(12)((1+.04/12)^{60} - 1)/(.04) \;\approx\; \$19{,}890$$

Tre made 60 deposits of \$300, his deposits total \$18,000 since 60(\$300) = \$18,000. The total interest earned is \$1,889 which is the difference between the ending balance \$19,889 and the total deposits of \$18,000.

$$\$19{,}890 \;-\; \$18{,}000 \;=\; \$1890$$

Instead of making regular deposit each month which are compounded monthly for a specified period of time, consider the reverse of this process which is an annuity. For an annuity a large amount is deposited initially at a compounded monthly rate which provides for regular monthly withdrawals for a specified period of time until the original deposited amounts runs out. For an annuity the future value of the large amount initially deposited must equal the future value of the sum of all the monthly withdrawals.

Formula An **annuity** with an initial deposit P which is compounded monthly at an annual interest rate r and distributes monthly withdrawals M for a period of t years is determined by the following formula.

Future value of P = Future value of all the M monthly withdrawals

$$P(1+r/12)^{12t} = (M)(12)((1+r/12)^{12t}-1)/(r)$$

Example 6 Kayla has \$200,000 for retirement and purchases an annuity so that she will receive a monthly income for 10 years. If the annuity has an annual interest rate of 5% which is compounded monthly, find the monthly payments that Kayla will receive.

Insert the following into the annuity formula: $P = 200{,}000$ $r = .05$ and $t = 10$
Round the left side of the formula to nearest cent and the right side to 6 significant digits and solve for monthly withdrawals M. This \$200,000 annuity will result in monthly withdrawals of \$2121.31 for 10 years.

$$P(1+r/12)^{12t} = (M)(12)((1+r/12)^{12t}-1)/(r)$$

$$200{,}000(1+.05/12)^{120} = (M)(12)((1+.05/12)^{120}-1)/(.05)$$

nearest cent *6 sig digits*

$$329{,}401.90 = 155.282M \qquad \text{Divide both sides by 155.282}$$

$$2121.31 = M$$

The same formula used for an annuity can be applied to mortgages. For a mortgage, instead of the individual providing the large initial amount this is the loan amount provided by a lender and instead of the individual receiving monthly income they have to make monthly payments.

Formula A **mortgage loan** of P dollars which is compounded monthly an annual interest rate r would require monthly payments M for a period of t years is determined by the following formula.

$$\text{Future value of } P = \text{Future value of all the } M \text{ monthly payments}$$

$$P(1+r/12)^{12t} = (M)(12)((1+r/12)^{12t}-1)/(r)$$

Example 7 Kim purchases a home for \$250,000 and makes a down payment of 5% and finances the rest with a 30 year loan at 3.45% annual interest rate which is compounded monthly. Find the monthly payments.

The down payment is 5%(\$250,000) = \$12,500

The loan amount \$237,500 is the difference of loan amount and the down payment.

$$\$250{,}000 - \$12{,}500 = \$237{,}500$$

Insert the following into the mortgage formula: $P = 237{,}500$ $r = .0345$ and $t = 30$
Round the left side of the formula to nearest cent and the right side to 6 significant digits and solve for monthly payments M. This \$237,500 loan will result in monthly payments of \$1059.86 for 30 years.

$$P(1+r/12)^{12t} = (M)(12)((1+r/12)^{12t}-1)/(r)$$

$$237{,}500(1+.0345/12)^{360} = (M)(12)((1+.0345/12)^{360}-1)/(.0345)$$

nearest cent *6 sig digits*

$$667{,}595.64 = 629.888\,M \qquad \text{Divide both sides by } 629.888$$

$$1059.86 = M$$

Since Kim will make 360 monthly payments of \$1059.86 during the lifetime of the loan, the total of the all the loan payments is \$381,550

$$360(\$1059.86) \approx \$381{,}550$$

The total interest paid during the lifetime of the loan \$144,050 is the difference of total loan payments of \$381,550 and the original loan amount of \$237,500.

$$\$381{,}550 - \$237{,}500 = \$144{,}050$$

Example 8 Randy purchases a car for \$25,000 and makes a down payment of \$5000 and finances the rest with a 5 year loan at 6% annual interest rate which is compounded monthly. Find the monthly payments.

The loan amount \$20,000 is the difference of loan amount and the down payment.

$$\$25{,}000 \; - \; \$5000 \; = \; \$20{,}000$$

Insert the following into the mortgage formula: $P = 20{,}000$ $r = ..06$ and $t = 5$
Round the left side of the formula to nearest cent and the right side to 6 significant digits and solve for monthly payments *P*. This \$20,000 loan will result in monthly payments of \$386.66 for 5 years.

$$P(1 + r/12)^{12t} \quad = \quad (M)(12)((1+r/12)^{12t} - 1)/(r)$$

$$20{,}000(1 + .06/12)^{60} \quad = \quad (M)(12)((1+.06/12)^{60} - 1)/(.06)$$

nearest cent *6 sig digits*

$$26{,}977.03 \quad = \quad 69.7700\,M \qquad \text{Divide both sides by } 69.7700$$

$$386.66 \quad = \quad M$$

Since Randy will make 60 payments of \$386.66 during the lifetime of the loan, the total of all the loan payments is \$23,200.

$$60(\$386.66) \; \approx \; \$23{,}200$$

To find the total interest paid during the lifetime of the loan \$3200 find the difference of total loan payments \$23,200 and the original loan amount \$20,000.

$$\$23{,}200 \; - \; \$20{,}000 \; = \; \$3{,}200$$

Exercises 3.7

1-4. Use the summation formula listed below to answer the following problems.

$$\sum_{0}^{k-1} b(a)^x \quad = \quad b((a)^k - 1)/(a - 1)$$

1. First 10 terms of geometric sequence $f(t) = 80(1.10)^t$

2. First 18 terms of geometric sequence $f(t) = 100(1.08)^t$

3. First 36 terms of geometric sequence $f(t) = 50(1.02)^t$

4. First 50 terms of geometric sequence $f(t) = 25(1.015)^t$

5-8. Use the future value formula listed below to answer the following problems.

$$\sum_{0}^{12t-1} M(1+r/12)^x \quad = \quad (M)(12)((1+r/12)^{12t} - 1)/(r)$$

5. Sal invests \$200 per month for four years at an annual interest rate of 3% which is compounded monthly. Find the balance in the account after four years.

6. To save money for a down payment on a house, Tim saves \$175 per month for five years at an annual interest rate of 4% which is compounded monthly. Find how much Tim has saved in this account after five years.

7. Rita plans to invest \$150 in a mutual fund for ten year and she hopes that the account returns the historical average of 7% per year compounded monthly during this time period. What is Rita's expected balance after 10 years.

8. To save money for holiday presents for her family, Rita invests \$50 per month in account in an account with an annual interest rate of 2% which is compounded monthly. How much is this account 9 months later?

9-12. Use the formula below to solve the following annuity or mortgage problem. Round the left side of this formula to nearest cent and the right side to 6 significant digits.

$$P(1+r/12)^{12t} \;=\; (M)(12)((1+r/12)^{12t}-1)/(r)$$

9. Earl has $150,000 for retirement and purchases an annuity so that he will receive a monthly income for twenty years. If the annuity has an annual interest rate of 4% which is compounded monthly, find the monthly withdrawals M that Earl will receive.

10. Tyler receives an inheritance of $50,000 and purchases an annuity to receive a monthly income for five years. If the annuity has an annual interest rate of 3.5% which is compounded monthly, find the monthly withdrawals M that Tyler will receive.

11. Antonio purchases a new car for $16,000 makes a down payment of $2000 and finances the rest with a 4 year loan at 7% annual interest rate which is compounded monthly. Find the monthly payments M. Also find how much in total loan payments and the total interest paid during the life of the loan.

12. Cristal purchases a home for $300,000 makes a down payment of 3% and finances the rest with a 30 year loan at 3.95% annual interest rate which is compounded monthly. Find the monthly payments M. Also find how much in total loan payments and the total interest paid during the life of the loan.

13-16. Use the formula below to solve the following annuity or mortgage problem. Round the left side of this formula to 6 significant digits and the right side to nearest cent.

$$P(1+r/12)^{12t} \;=\; (M)(12)((1+r/12)^{12t}-1)/(r)$$

13. Find the initial amount P needed to purchase an annuity with annual interest rate of 2% which is compounded monthly that will provide monthly withdrawals of $800 for ten years.

14. Find the initial amount P needed to purchase an annuity with annual interest rate of 3.7% which is compounded monthly that will provide monthly withdrawals of $2000 for five years.

15. Ronnel intends to purchase a home with a 30 year loan at 3.75% annual interest rate which is compounded monthly. If he can afford monthly payments of $1500, what is the maximum loan amount P that he can obtain?

16. Tia intends to purchase a new car with a 3 year loan at 6.5% annual interest rate which is compounded monthly. If she can afford monthly payments of $300, what is the maximum loan amount P that she can obtain?

Section 3.8: The TI Calculator and Exponential Regression

Finding the exponential regression curve using the TI graphing calculator is similar to finding the regression line, but we chose a different calculator function.

Step 1: Enter the Data into two lists.
Step 2: Graph a scatter plot to visually determine if a linear or exponential model should be made.
Step 3: Calculate the regression line/ curve by pressing

STAT

➢ (To highlight CALC)

Scroll down to find:

LinReg(ax+b) (Select if the scatterplot appears linear)

ExpReg (Select if the Scatterplot appears exponential)

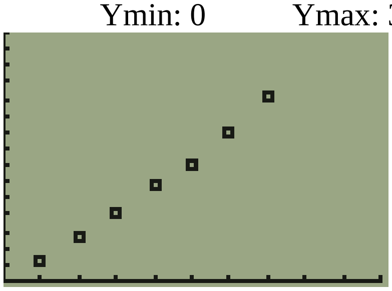
XList: Enter the list you entered the independent data into
YList: Enter the list you entered the dependent data into
FreqList: Leave blank
Store RegEQ: Leave blank
Calculate (highlight this and press ENTER)

Example 1:
Data

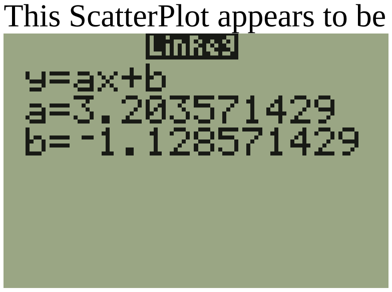
X	1	2	3	4	5	6	7
Y	2. 4	5.5	8.2	11.7	14	17.9	22.1

Enter X into L1
Enter Y into L2
Turn on the Scatterplot and set the window. I choose the following for my window:

Xmin:0 Xmax: 10 Xscl: 1
Ymin: 0 Ymax: 30 Yscl: 2



This ScatterPlot appears to be linear, so use the LinReg(ax+b) command:



So the best fit linear equation is: y = 3.2036x – 1.1286 (rounded to 4 decimal places)

Example 2:

Data

X	1	2	3	4	5	6	7
Y	2. 4	5.5	10.2	23	50.5	97.3	210.6

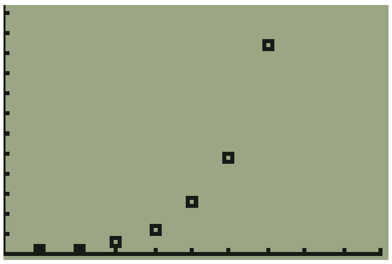
Enter X into L1

Enter Y into L2

Turn on the Scatterplot and set the window. I choose the following for my window:

Xmin:0 Xmax: 10 Xscl: 1

Ymin: 0 Ymax: 250 Yscl: 20



This ScatterPlot appears to be Exponential, so use the ExpReg command:

The best fit exponential curve is: $y = 1.1733 * (2.0996)^x$

(rounded to 4 decimal places)

We can tell from the equation that the y values are approximately doubling as x increases by 1.

Section 3.8 Homework

For each question 1-8: Determine if a line, exponential, or neither is the best model. If the best model is either linear or exponential then find the best fit equation.

1)

X	0	0.9	2	3.1	4.2	5.1
Y	5.1	15.2	40.7	130.3	400.2	1200

2)

X	1	1.9	3.2	4.1	5.5
Y	5.1	15.2	27.3	35.8	49.6

3)

X	1	2	3	4	5	6	7	8	9
Y	100	49	23.8	12.6	8	4.3	2.1	1.4	0.8

4)

X	1	2	3	4	5	6	7
Y	5	10	20	30	20	10	5

5)

X	0	1.1	3.2	5.5	7.9	8.6
Y	0.5	1.1	2.5	5.1	10.2	22.7

6)

X	0	1	3	4	4	5	6	8	8
Y	100	94.5	85.9	80.3	77.9	75.9	70.2	60.2	58.3

7)

X	1	3	7	9	13
Y	7	15.9	32.3	44.5	60.2

8)

X	1	2	3	4	5	6	7
Y	2	3.2	7.9	15.5	32.9	70	155.9